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A THEORY OF THE DUCTED PROPELLER WITH A FINITE NUMBER OF BLADES

by

WILLIAM B. MORGAN

Under Contract Number N-onr-222(30)

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Table of Contents

Abstract

A theory for the ducted propeller is developed which is based on a linearized annular airfoil theory and a lifting-line propeller theory. The fluid is assumed to be inviscid and incompressible and the free-stream velocity to be axisymmetric. As with propeller theory it is not possible to obtain a solution in explicit form so a process of iteration is used.

The flow field of the annular airfoil is represented by a distribution of ring vortices and ring sources on a cylinder and where necessary a trailing vortex system. This approach allows the airfoil section to have an arbitrary shape although the annular airfoil itself is assumed to be axisymmetric. The ring source strength is shown to be a function of only the duct thickness while the ring vortex strength is a function of camber, thickness and the radial velocity induced on the cylinder by the propeller and duct trailing vortex system. In the presence of the propeller two coupled singular integral equations are derived for the vortex strength which are reduced to two coupled Fredholm equations of the second kind. (If the propeller is not present only one integral equation is obtained.)

The flow field of the propeller is represented by a lifting line and a helicoidal trailing vortex system. This allows the propeller to have a finite number of blades and an arbitrary distribution of circulation. By this approach the propeller problem essentially reduces to the propeller by itself with the

inclusion of velocity components from the duct and hub. Consequently, it reduces essentially to the propeller problem solved by Lerbs.

The hub is treated by slender body theory which allows it to have an arbitrary axisymmetric shape. One consequence of using this theory is that the hub induces no tangential velocities.

Notation

a	duct chord
a_ℓ	axial distance between leading edge of duct and propeller
a_t	axial distance between trailing edge of duct and propeller
b	number of blades
C_{Tsi}	thrust coefficient, equations (2.7-9) and (4.4-4)
C_{Psi}	power coefficient, equation (4.4-6)
$c_1(z)$	mean line ordinate of the duct section measured from the nose-tail line
$E(k)$	complete elliptic integral of the second kind
G_s	nondimensionalized circulation distribution of the propeller, equation (4.3-11)
h	$(a/2R_d)$ chord-diameter ratio of the duct
$\vec{i},\vec{j},\vec{k}$	unit vectors
$I_n(y)$	modified Bessel function of the first kind
$K_n(y)$	modified Bessel function of the second kind
k	modulus of the elliptic integrals
$L_n(y)$	modified Struve function
Q	propeller torque
q	ring source strength
q_h	hub source strength
R_d	duct radius
R_p	propeller radius
r, φ, ζ	cylindrical coordinates
r_o	radius at which the vortex is shed from the propeller blade

$s(z)$	half thickness ordinate of the duct section
T	thrust
V_s	ship speed
w_a	axial component of induced velocity
w_o	local axial velocity
w_r	radial component of induced velocity
w_t	tangential component of induced velocity
w_x	local wake fraction
$\hat{x}, y, \xi$	rectangular coordinates
x, φ, z	nondimensionalized cylindrical coordinates
x	radial coordinate nondimensionalized by the propeller radius
x_o	nondimensionalized radius at which a vortex is shed from the propeller blade
z	axial coordinate nondimensionalized by the duct chord
z_p	axial coordinate nondimensionalized by the propeller radius
$\bar{z}$	$z - a_t$
α	angle of attack of a duct section
α_{id}	ideal angle of attack of a duct section
α_r	relative angle between free-stream velocity and duct
β	propeller advance angle
β_d	duct advance angle
β_i	propeller hydrodynamic pitch angle
γ	ring vortex strength
Γ	circulation

ξ	axial coordinate
η_i	ideal efficiency, equation (4.4-7)
λ	advance coefficient, equation (4.6-2)
λ_s	advance coefficient, equation (4.4-3)
ρ	mass density of fluid
ψ	stream function
ω_o	angular velocity

Subscripts

d	duct
h	hub
p	propeller
q	ring source
γ	ring vortex
$\frac{\partial\gamma}{\partial\varphi}$	trailing vortex system of the vortex cylinder

Note: Many functions are defined in the text.

A. THEORY OF THE DUCTED PROPELLER WITH A FINITE NUMBER OF BLADES

I. INTRODUCTION

The name "ducted propeller" means a propeller-annular airfoil combination acting as a propulsion unit. The annular airfoil or duct can be used either to accelerate the flow at the propeller (Kort nozzle)[1] or to deaccelerate the flow. In the first type of flow the ducted propeller is used where a propeller alone would be heavily loaded. The duct accelerates the flow at the propeller and thus the propeller operates at a more favorable loading, in addition, the duct itself will generally produce a positive thrust. In the case of the duct which deaccelerates the flow, the annular airfoil is used to increase the static pressure at the propeller and thus delay cavitation on ship propellers or decrease compressibility effects on aircraft propellers.

Most of the work on the ducted propeller of the "Kort nozzle" type has been done in Europe. Well known are the experimental results of Van Manen [2,3,4,5] for a systematic series of "Kort nozzles." A series for low-pitched three-bladed propellers in a duct has also been given by Nakonechny[6]. Most theoretical approaches have been restricted to representing the duct by a distribution of ring vortices along a cylinder[7] and the enclosed propeller by a distribution of sources over the propeller disc.

1 - References found on page 146

One of the first papers on the theory of the ducted propeller is that of Horn[8] in which he uses the work of Dickmann[7] for representing the nozzle and considers the propeller as free-running. This procedure did not lead to a design method. In 1950 Horn and Amtsburg[9] developed a design procedure based on representing the duct by a vortex distribution and the propeller by a sink distribution. Later (1955) Dickmann and Weissinger[10] considered the ducted propeller as a propulsion unit and in the optimum case represented the duct by ring vortices with effect of the propeller taken into consideration by momentum theory. In 1959 Gutsche[11] developed correction factors from the propeller in a long tube and then used simple jet theory.

The foregoing described results have in general been restricted to ducts which accelerate the flow at the propeller although the theory developed is in principle applicable to the deaccelerating duct. Lerbs[12] has applied the theory, i.e. representing the duct by a vortex distribution and the propeller by a sink distribution, specifically to deaccelerating flow in the duct. Kuchemann and Weber[13] have considered the ducted propeller in general but only with simple momentum theory.

From a review of the literature it is apparent that the problem of the ducted propeller with a finite number of blades and arbitrary distribution of circulation has not been developed previously* but that such a theory is necessary for the adequate

*After the work in this report was completed, Reference [41] was received which considers a lightly loaded propeller with a finite number of blades in a duct of zero thickness.

design of propellers operating in a duct. This paper presents such a theory in which it is assumed that the nozzle flow field can be represented by a linearized theory and the propeller by the lifting line theory. As usual for potential flow problems a number of assumptions are made about the fluid as well as about the flow field. For this problem these are:

1. The fluid is inviscid and incompressible and no separation occurs on the duct.

2. Body forces, such a gravity, may be neglected.

3. The free-stream flow is axisymmetric and steady with respect to a coordinate system attached to the propeller. This allows a radial variation in velocity and implies that the coordinate system is rotating with the propeller. It causes no loss in generality to assume the duct is also rotating since the duct by itself at zero incidence induces no tangential velocity.

4. The annular airfoil is axisymmetric and of finite length.

5. The thickness and camber-distribution of the annular airfoil section can be expanded in a Fourier series with respect to the axial coordinate. This assumption offers no restriction to streamline shapes.

6. The linearized flow around the annular airfoil can be represented mathematically by a distribution of ring vortices and ring sources along a cylinder of diameter R_d.

7. The trailing vortex system from the duct has a constant diameter R_d and extends from the duct to infinity.

8. The influence of all induced velocities on the shape of the trailing vortex system from the duct is neglected. This implies that the pitch angle of all the free vortex lines from the duct is the same and equal to

$$\tan\beta_i = \frac{w_i(R_d)}{\omega_0 R_d} = \frac{V_s(1-w_T)}{\omega_0 R_d}$$

9. The propeller flow field can be represented by a lifting line and trailing vortices, i.e. a horse-shoe vortex system. The trailing vortex system is directed along helical stream lines trailing aft from the propeller blades. Each vortex is of constant pitch and lies on a cylinder of constant diameter. This implies that the contraction of the slip-stream is ignored.

Using these assumptions the linearized boundary conditions on the duct are derived. The annular airfoil is first considered by itself and its flow field represented by ring sources and ring vortices. The strength of each is chosen so that the boundary conditions are satisfied. With the strength of the vortex and source distribution known, the entire flow field of the duct can be derived. The propeller with a finite hub is then added to the flow field and the interaction effects determined. The solution to the problem of the combination of the propeller and duct reduces to a process of iteration.

II. LINEARIZED THEORY OF THE ANNULAR AIRFOIL

II.1 Previous Theories

The theory of the annular airfoil has been discussed by numerous investigators but in most cases was not developed sufficiently for ducts of arbitrary section shape. The first theoretical discussion of the annular airfoil appears to be that of Dickmann[10] who represented the annular airfoil in uniform axial flow by a distribution of ring vortices. This is equivalent in thin wing theory to representing an airfoil by a distribution of vortices only, and, thus, the thickness of the foil is neglected. In linearized two-dimensional wing theory neglecting the thickness is justifiable for obtaining the lift but not for the pressure and velocity distribution. In annular airfoil theory it would be expected, because of the interference effects, that the thickness plays a more important role than in linearized wing theory.

The solution for the induced velocities from a single vortex ring was given in the form of elliptic integrals by Lamb.[14] By arranging ring vortices of varying strength along a cylinder Dickmann represented the resulting integral of elliptic integrals by a Fourier series. Tabulated coefficients for determining the velocity distribution of a vortex ring and vortex cylinder are found in Kuchemann and Weber[13] and a more complete theoretical development of singularities useful for this problem is found in a report by Meyerhoff and Finkelstein[15].

Stewart,[16] independently of Dickmann, derived the induced velocities of an annular airfoil, again represented by a vortex cylinder, using the vector potential. He was able to represent the velocity components by integrals of a product of modified Bessel functions.

Some work has also been done on flow about thick annular airfoils and foils at an angle of attack. Specifically Kuchemann[17] considered annular airfoils of finite thickness without circulation by a distribution of source and sink rings and later Kuchemann and Weber[18] considered annular foils of finite thickness with circulation but infinite length. In either case the theory was not adapted to foils of arbitrary shape but the shape and velocity distribution calculated for an assumed distribution of sources and sinks.

Weissinger[19] has discussed the flow field about annular airfoils with zero thickness operating at an angle of incidence. To represent the flow mathematically he uses a distribution of ring vortices along the duct whose strength, $\gamma(\varphi, z)$, at a point on the ring is dependent on the angular position as well as the axial. Since in this case there are free vortices in addition to the bound vortices, he uses, in addition, a system of vortices of strength $\frac{1}{R_d}\frac{\partial \gamma}{\partial \varphi}$ trailing from the cylinder. An integral equation for the vortex distribution results from this analysis which is solved approximately. Weissinger[20] later included the effect of finite thickness by using a distribution

of ring sources. An approach which follows Weissinger's work very closely is that of Bagley, Kirby and Marcer[21]. In their work use is made of standard ring vortex distributions which were tabulated by Kuchemann and Weber[13]. Since this method is restricted to satisfying the boundary condition at an arbitrary number of points along the chord, (maximum of five) it gives only an approximate solution. Their consideration of the annular airfoil at an angle of attack is similar to Weissinger's.

Recently Pivko[22] considered annular airfoils with thick symmetrical sections but his work is only applicable to nozzle length-diameter ratios of much less than one. This restriction comes from the fact that he assumed a symmetrical section could be represented by a distribution of sources and sinks. Because of the interference effects, however, a vortex distribution must also be used in addition to the source-sink distribution. Pivko[23] has also considered thick cambered annular airfoils operating at an angle of attack and included the effect of the propeller by a sink disc. In general he makes use of the velocity coefficients given by Kuchemann and Weber[13] and superimposes the velocity of each effect. This theory is not readily adaptable to sections of arbitrary shape.

In addition to the preceding work Malavard[24] has considered the pressure distribution on annular airfoils with and without thickness using electrical analogy and Hacques[25] has considered

the problem of designing for a given pressure distribution also by electrical analogy.

In the next section the linearization of the boundary conditions on the duct is considered. When linearizing inviscid, incompressible fluid problems, it is the boundary conditions which are linearized as the equation of continuity, which reduces to Laplace's equation in this case, is linear.

II.2 Linearized Boundary Conditions

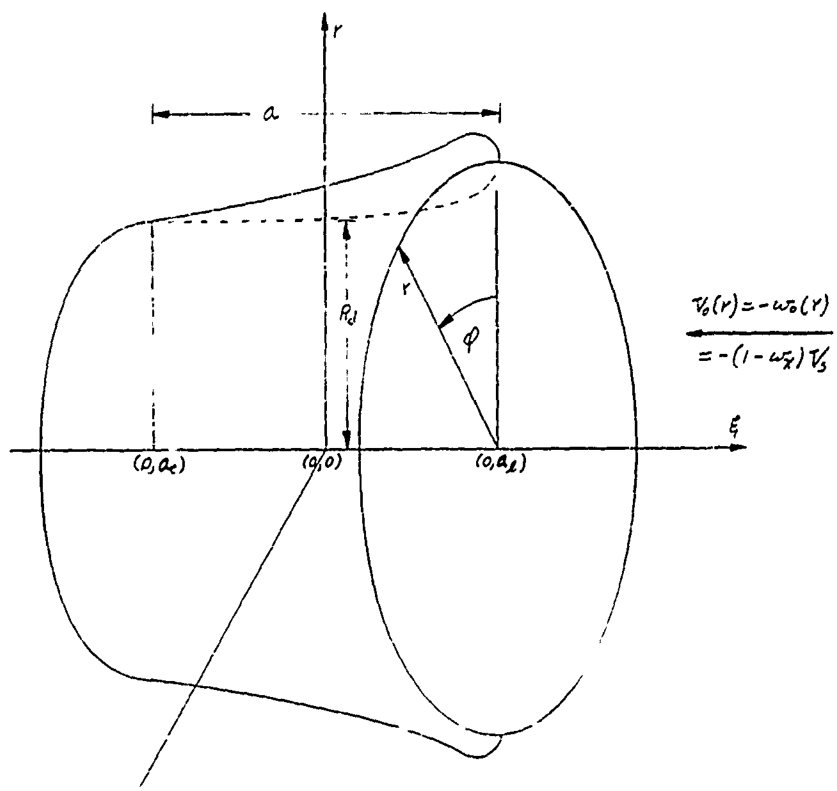

Figure 1. The annular airfoil coordinate system

The coordinate system which is adopted is a cylindrical system (r , φ , ξ) with the zero axial coordinate (ξ) located at the propeller blade center line and the free-stream flow is from right to left (see Figure 1). The annular airfoil is assumed axisymmetric and has a length (a). Any radius on the foil can be chosen as the reference radius (R_d) but since this reference radius is the radius of the cylinder along which the vortices and sources are distributed it would seem reasonable to use some sort of an average. For convenience, but somewhat arbitrarily, the reference radius will be taken as the inside radius of the annular airfoil at the propeller.

The section shape is assumed to be delineated by the outside of the annular airfoil u(ξ) and by the inside b(ξ) as shown in figure 2.

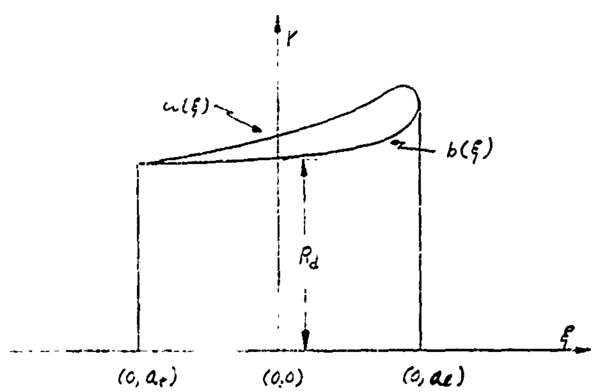

Figure 2. The annular airfoil section

Using the above notation the mean line of the foil as measured from the ξ -axis (r=0) is

$$c(\xi) = \frac{1}{2}\left[u(\xi) + b(\xi)\right] \qquad (2.2\text{-}1)$$

and the half thickness ordinate is

$$s(\xi)=\tfrac{1}{2}\left[u(\xi)-b(\xi)\right] \tag{2.2-2}$$

The outer surface in terms of the mean line ordinate, or camber, $c(\xi)$ and the half thickness ordinate $s(\xi)$ is then given by

$$r=u(\xi)=c(\xi)+s(\xi) \tag{2.2-3}$$

and the inner surface by

$$r=b(\xi)=c(\xi)-s(\xi) \tag{2.2-4}$$

If it is assumed that the mean line deviates little from the cylinder of radius R_d and length a and that the thickness is small, then the camber and the thickness can be expanded in terms of a nondimensional perturbation parameter ϵ.

$$r=u(\xi;\epsilon)=[c(\xi;\epsilon)+s(\xi;\epsilon)]=R_d+\epsilon[c^{(1)}(\xi)+s^{(1)}(\xi)]+\epsilon^2[c^{(2)}(\xi)+s^{(2)}(\xi)]+\cdots$$
$$\simeq R_d+\epsilon[c^{(1)}(\xi)+s^{(1)}(\xi)] \tag{2.2-5}$$

$$r=b(\xi;\epsilon)=[c(\xi;\epsilon)-s(\xi;\epsilon)]=R_d+\epsilon[c^{(1)}(\xi)-s^{(1)}(\xi)]+\epsilon^2[c^{(2)}(\xi)-s^{(2)}(\xi)]+\cdots$$
$$\simeq R_d+\epsilon[c^{(1)}(\xi)-s^{(1)}(\xi)] \tag{2.2-6}$$

As $\epsilon \to 0$ the problem reduces to that of a thin circular cylinder of constant diameter.

Since the problem is axisymmetric and linear, a total stream function can be written for the flow in terms of the perturbation parameter ϵ.

$$\Psi(r,\xi;\epsilon) = -\int w_0(r)\,r\,dr + \psi(r,\xi;\epsilon) = -\int w_0(r)\,r\,dr + \epsilon\,\psi^{(1)}(r,\xi)$$
$$+\epsilon^2\,\psi^{(2)}(r,\xi) + \cdots \tag{2.2-7}$$

As $\epsilon \to 0$ the stream function reduces to that for the free-stream velocity alone. From equations (2.2-5), (2.2-6) and (2.2-7) the linearized boundary conditions are developed. First there is the kinematic boundary condition, i.e. on the surface of the body $\vec{V}\cdot\vec{n}\big|_{body} = 0$. As a first approximation the equation for the outer surface of the ring is obtained from equation (2.2-5).

$$F(r,\xi) = R_d + \epsilon\left[c^{(1)}(\xi) + s^{(1)}(\xi)\right] - r = 0 \tag{2.2-8}$$

and for the inner surface of the ring from equation (2.2-6)

$$F(r,\xi) = R_d + \epsilon\left[c^{(1)}(\xi) - s^{(1)}(\xi)\right] - r = 0 \tag{2.2-9}$$

The velocity $\vec{V}$ is obtained from the stream function and in terms of its components is

$$\begin{aligned} w_a(r,\xi) &= \frac{1}{r}\frac{\partial\Psi}{\partial r} = \frac{1}{r}\Psi_r \\ w_r(r,\xi) &= -\frac{1}{r}\frac{\partial\Psi}{\partial\xi} = -\frac{1}{r}\Psi_\xi \end{aligned} \tag{2.2-10}$$

In terms of these velocity components and the normal to the surface, the kinematic boundary condition becomes

$$\vec{V}\cdot\vec{n}\big|_{body} = \frac{w_r F_r(r,\xi) + w_a F_\xi(r,\xi)}{\left[F_r^2(r,\xi) + F_\xi^2(r,\xi)\right]^{1/2}} = 0 \tag{2.2-11}$$

From equations (2.2-8) and (2.2-9), it follows that $F_r(r,\xi)$

and $F_\xi\ (r,\xi)$ are, on the outside of the annular airfoil

$$F_r(r,\xi) = -1$$
$$F_\xi(r,\xi) = \epsilon[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)] \tag{2.2-12}$$

and on the inside

$$F_r(r,\xi) = -1$$
$$F_\xi(r,\xi) = \epsilon[c^{(1)\prime}(\xi) - s^{(1)\prime}(\xi)] \tag{2.2-13}$$

With these values for the normal to the surface, it follows from equation (2.2-10) and (2.2-11), after multiplying through by $r\sqrt{F_r^2(r,\xi) + F_\xi^2(r,\xi)}$, that the kinematic boundary condition on the outside of the ring is

$$\Psi_r \cdot \epsilon[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)] + \Psi_\xi = 0 \tag{2.2-14}$$

and on the inside of the ring is

$$\Psi \cdot \epsilon[c^{(1)\prime}(\xi) - s^{(1)\prime}(\xi)] + \Psi_\xi = 0 \tag{2.2-15}$$

The stream function Ψ is given by equation (2.2-7) and substituting this value into equation (2.2-14), the boundary condition on the outside of the ring is obtained in terms of the perturbation parameter ϵ.

$$\Big\{-(R_d + \epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)]\omega_0(R_d + \epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)])$$
$$+\psi_r(\epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)], \xi; \epsilon)\Big\}\, \epsilon[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)]$$
$$+\psi_\xi(R_d + \epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)], \xi; \epsilon) = 0 \tag{2.2-16}$$

The stream function is a function of $\epsilon[c^{(1)}(z) + s^{(1)}(z)]$ and is next expanded in Taylor's series in terms of this parameter. Equation (2-16) then becomes

$$\left\{-R_d w_0(R_d) + \cdots + \psi_r(R_d+0,\xi;\epsilon) + \psi_{rr}(R_d+0,\xi;\epsilon)\,\epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)] + \cdots\right\}$$

$$\cdot\,\epsilon[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)] + \left\{\psi_\xi(R_d+0,\xi;\epsilon) + \psi_{\xi r}(R_d+0,\xi;\epsilon)\,\epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)] + \cdots\right\}. = 0 \qquad (2.2-17)$$

The stream function $\psi(R_d+0,\xi\,;\epsilon)$ can also be expanded in terms of ϵ , this equation then becomes

$$\left\{-R_d w_0(R_d) + \cdots + \epsilon\,\psi_r^{(1)}(R_d+0,\xi) + \epsilon^2\psi_r^{(2)} + \cdots + [\epsilon\,\psi_{rr}^{(1)} + \epsilon^2\psi_{rr}^{(2)} + \cdots]\right.$$

$$\left.\cdot\,\epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)] + \cdots\right\}\epsilon[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)]$$

$$+\left\{\epsilon\,\psi_\xi^{(1)} + \epsilon^2\psi_\xi^{(2)} + \cdots [\epsilon\,\psi_{\xi r}^{(1)} + \epsilon^2\psi_{\xi r}^{(2)} + \cdots]\,\epsilon[c^{(1)}(\xi) + s^{(1)}(\xi)] + \cdots\right\} = 0$$

(2-18)

Collecting terms of the same order the previous equation can be written as

$$\epsilon\left\{-R_d w_0(R_d)[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)] + \psi_\xi^{(1)}(R_d+0,\xi)\right\}$$

$$+\,\epsilon^2\left\{\psi_r^{(1)}(R_d+0,\xi)[c^{(1)\prime}(\xi) + s^{(1)\prime}(\xi)] + \psi_\xi^{(2)}(R_d+0,\xi) + [w_0(R_d) + R_d w_{0r}(R_d)\right.$$

$$\left.+\,\psi_{\xi r}^{(1)}(R_d+0,\xi)]\cdot[c^{(1)}(\xi) + s^{(1)}(\xi)]\right\} + \epsilon^3\{\quad\} + \cdots = 0$$

(2-19)

The first approximation is obtained by neglecting terms involving powers of ϵ greater than one. Derivation of the

boundary condition for the inside of the annular airfoil follows in similar manner. Considering both sides of the ring, the first approximation for the kinematic boundary condition is

$$\psi_{\xi}^{(1)}(R_d \pm 0, \xi) = R_d\, w_0(R_d)\left[c^{(1)\prime}(\xi) \pm s^{(1)\prime}(\xi)\right] \qquad (2.2\text{-}20)$$

The + sign refers to the outside of the annular airfoil and the - sign to the inside. This equation has only to be satisfied on the circular cylinder of diameter (R_d) and length (a).

In addition to the foregoing boundary condition, the Kutta condition must be satisfied at the trailing edge of the ring. This means essentially that a stagnation point must occur at the trailing edge of the ring and for this the radial velocity at this point must be zero. Consequently at the ring trailing edge the stream function must satisfy the following equation,

$$w_r(R_d \pm 0, a_t) = 0 = \psi_{\xi}^{(1)}(R_d \pm 0, a_t) \qquad (2.2\text{-}21)$$

Since it has been assumed as a first approximation that $\psi = \psi^{(1)}$ and similarly for $c^{(1)}(\xi)$, and $s^{(1)}(\xi)$, the superscripts in the last equations can be dropped and the boundary conditions can then be written as

$$\psi_{\xi}(R_d \pm 0, \xi) = R_d\, w_0(R_d)\left[c'(\xi) \pm s'(\xi)\right] \quad \text{on} \quad a_t \le \xi \le a_l$$

$$\psi_{\xi}(R_d \pm 0, a_t) = 0 \qquad (2.2\text{-}22)$$

For convenience the radial velocities will be nondimensionalized by the free-stream velocity if this velocity is

uniform or by the ship velocity V_s if the flow is axisymmetric. The axial coordinate will also be nondimensionalized by the annular airfoil chord a and the radial coordinate by the propeller radius R_p, i.e.

$$z = \frac{\text{axial coordinate}}{\text{duct chord}} = \frac{\xi}{a}$$

$$x = \frac{\text{radial coordinate}}{\text{propeller radius}} = \frac{r}{R_p} \qquad (2.2\text{-}23)$$

$$x_d = \frac{\text{duct radius}}{\text{propeller radius}} = \frac{R_d}{R_p}$$

; if the duct diameter is equal the propeller diameter then $x_d=1$

The following notation will also be introduced which is consistent with common usage in naval architecture.

$$(1 - w_x) = \frac{w_0(x)}{V_s} \qquad (2.2\text{-}24)$$

If the velocity is uniform, i.e. independent of radius, instead of the ship speed, the free-stream velocity is used for nondimensionalizing and the wake fraction is unity, i.e.

$$(1 - w_x) = \frac{w_0}{w_0} = 1$$

In the definition of the half-thickness ordinate, equation (2.2-2) and mean line ordinate, equation (2.2-1), the angle of attack of the section was not discussed. The mean line ordinate is measured from r=0 and it is convenient to decompose this ordinate into a part from camber and a part from angle of attack, see figure 3.

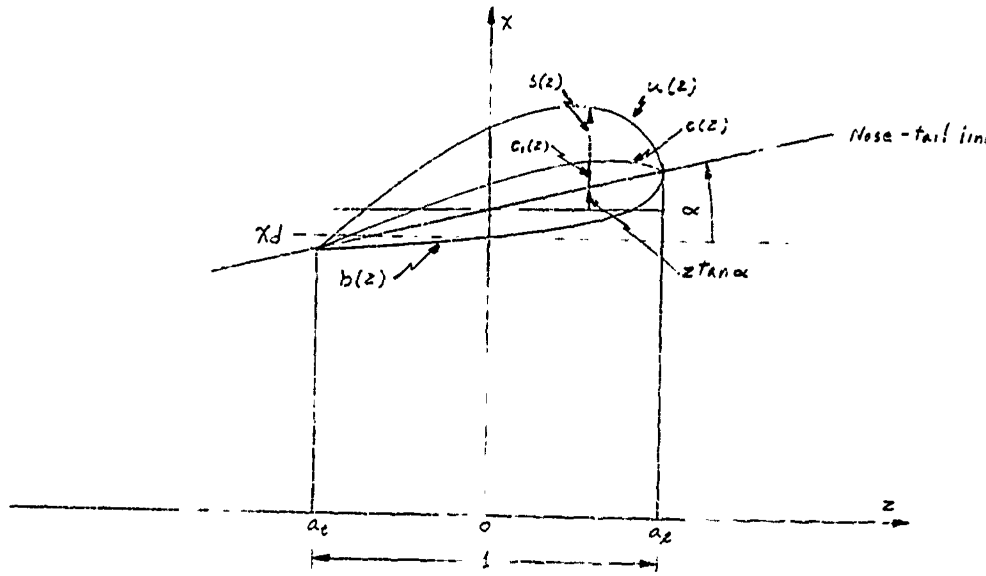

Figure 3. Delineation of the annular airfoil section

Using the notation given in Figure 3 the mean line ordinate c(z) can be expressed in the following form.

$$c(z) = \chi_d + s(0) + z\tan\alpha + c_1(z) - c_1(0) \tag{2.2-25}$$

It may often be assumed for practical purposes that $c_1(z)$ and s(z) may be measured perpendicularly to the nose-tail line. This is for convenience in delineating the camber and thickness, since normally the nose-tail line is used as a reference in describing section shapes, and implies that the angle α is small ($\alpha \simeq \tan\alpha$). If the angle α is too large for such an assumption then, although inconvenient, $c_1(z)$ and s(z) must be taken perpendicularly to the z-axis. For two-dimensional airfoils it would be expected that angles for which $\alpha \neq \tan\alpha$ would be outside the applicability of the linearized theory

but this may not be necessarily so for the sections of an annular airfoil.

If the foregoing equation for the camber is substitituted into the boundary condition, equation (2.2-22) and the coordinates and velocities are nondimensionalized as discussed, the boundary condition can be written as

$$\varphi_z(x_d \pm 0, z) = x_d(1 - w_{x_d})[c_1'(z) + \tan\alpha \pm s'(z)] \quad \text{on } a_t \leq z \leq a_\ell$$

$$\varphi_z(x_d \pm 0, a_t) = 0 \tag{2.2-26}$$

If a propeller is in the duct, then the radial velocity on the duct is a function of angular as well as axial position. In this case the singularities used to represent the duct must also be a function of both angular and axial position. Since the shape of the duct is assumed to be axisymmetric, (this assumption can be removed) the right-hand side of equation (2.2-26) is independent of the angle φ, however, the left-hand side includes all the radial velocities and can be dependent on the angular position, i.e. the following equations can be valid even though the right-hand side is independent of the angle φ.

$$\varphi_z(x_d \pm 0, \varphi, z) = x_d(1 - w_{x_d})[c_1'(z) + \tan\alpha \pm s'(z)] \tag{2.2-27}$$

II.3 Derivation of the Vortex and Source Distribution to Represent the Annular Airfoil at Zero Incidence

The boundary conditions just derived will be used to determine the strength of the ring vortex and source distributions

on the cylinder. These ring vortices, of elementary strength $\gamma(\varphi, z)$, and ring sources of elementary strength $q(\varphi, z)$, are used as a mathematical model to represent the flow around the annular airfoil. In order that the distribution of sources and sinks represents a closed body, it is required that there be no outflow of fluid from the source-sink distribution or that

$$\int_0^1 \int_0^{2\pi} q(\varphi, z)\, d\varphi\, dz = 0 \qquad (2.3\text{-}1)$$

If a propeller is in the duct, then the ring vortex strength is dependent on the angular position and a trailing vortex system exists behind the duct. This system has a strength of $\frac{1}{x_d} \frac{\partial \gamma}{\partial \varphi}$ and the induced velocities from this system must be added to that of the ring vortices and sources and those of the propeller. As discussed in the previous section the stream function occurring in the boundary condition is the total stream function for the flow, excluding that for the free stream, and since it is linear all the induced velocities from the various singularities are added linearly and the boundary condition on the duct is expressed by the following equation.

$$-[w_r(x_d, \varphi, z)]_\gamma - [w_r(x_d \pm 0, \varphi, z)]_q - [w_r(x_d, \varphi, z)]_{\frac{\partial \gamma}{\partial \varphi}}$$

$$-[w_r(x_d, \varphi, z)]_{p+h} = (1 - w_{x_d})\,[c_i'(z) + \tan\alpha \pm s'(z)] \qquad (2.3\text{-}2)$$

where

$[w_r(x_d,\varphi,z)]_\gamma$ = radial velocity induced on the duct by the ring vortex system and given by equation (A-16)

$[w_r(x_d\pm 0,\varphi,z)]_q$ = radial velocity induced on the duct by the ring source system and given by equation (B-8)

$[w_r(x_d,\varphi,z)]_{\frac{\partial\gamma}{\partial\varphi}}$ = radial velocity induced on the duct by the trailing vortex system and given by equation (C-13) or (D-8)

$[w_r(x_d,\varphi,z)]_{p+h}$ = radial velocity induced on the duct by the propeller including the hub.

Making the substitutions into equation (2.3-2) for the various velocity components, an equation is obtained for the vortex and source strength.

$$\frac{h}{2\pi}\int_0^1\oint_0^{2\pi}\frac{2h(\bar z - z')\cos(\varphi-\varphi')\,\gamma(\varphi',z')\,d\varphi'}{[4h^2(\bar z - z')^2+2-2\cos(\varphi-\varphi')]^{3/2}}\,dz'$$

$$+\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{[1-\cos(\varphi-\varphi')]\,q(\varphi',z')\,d\varphi'}{[4h^2(\bar z - z')^2+2-2\cos(\varphi-\varphi')]^{3/2}}\,dz' \quad \pm\frac{1}{2}q(\varphi,z)$$

$$+[w_r(x_d,\varphi,z)]_{\frac{\partial\gamma}{\partial\varphi}} = -(1-w_{x_d})[c_i'(z)+\tan\alpha \pm s'(z)] - [w_r(x_d,\varphi,z)]_{p+h} \qquad (2.3\text{-}3)$$

where $\bar z = z - a_t$

Since the integrals occurring in this equation have only one sign and since the radial velocity induced by the propeller on the duct does not change sign from one side of the duct to the other, it must be concluded that the $\pm$ signs go with the $\mp$ signs and hence the source-sink strength $q(\varphi, z)$ is given as follows:

$$q(\varphi,z) = -2(1-w_{x_d})\,s'(z) \qquad (2.3\text{-}4)$$

which implies that

$$q(\varphi, z) = q(z)$$

or that within the linearized theory the source distribution is independent of angle. If equation (2.3-4) is substituted into equation (2.3-5), a singular integral equation is obtained for the unknown circulation distribution which also includes the derivative of the circulation. To solve this equation for the circulation distribution requires a knowledge of the form of the radial velocity induced on the duct by the propeller and hub. Before deriving this velocity, it is convenient to consider the duct by itself, first at zero angle of attack and then at an angle of attack. In the first case the circulation distribution is independent of angle and the trailing vortex system does not exist. Utilizing equations (A-23) (B-11), and (2.3-4) the equation for the circulation distribution is obtained as:

$$\oint_0^1 \frac{\gamma(z')}{(\bar{z}-z')} k\left\{4h^2(\bar{z}-z')^2[K(k)-E(k)] - 2E(k)\right\}dz' \tag{2.3-5}$$
$$= 4\pi(1-w_{rd})[C_1'(z)+\tan\alpha] - 4h(1-w_{rd})\int_0^1 s'(z')\, k[K(k)-E(k)]dz'$$

where

$$k^2 = \frac{1}{h^2(z-z')^2+1}$$

$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta$ = complete elliptic integral of the first kind

$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta$ = complete elliptic integral of the second kind

$$h = \frac{a}{2R_d}$$

From this equation and equation (2.3-4) it can be seen that the vortex strength is a function of both camber and thickness distribution while the source strength is a function of only thickness. This differs from linearized wing theory where the vortex strength is a function of only camber. Equation (2.3-5) is a singular integral equation of the first kind and the solution of this equation will be discussed in the next section.

II.4 Reduction of the Integral Equation for the Vortex Distribution.

Equation (2.3-5) is a singular integral equation for the vortex distribution. For convenience this equation can be rewritten in the following form

$$\oint_0^1 g(\bar{z}-z')\frac{\gamma(z')}{(\bar{z}-z')}\,dz' = H(\bar{z}) \tag{2.4-1}$$

where

$$g(\bar{z}-z') = k\left\{4h^2(\bar{z}-z')^2[K(k)-E(k)] - 2E(k)\right\} \tag{2.4-2}$$

$$H(\bar{z}) = 4(1-w_{\gamma d})\left\{\pi[c_l'(\bar{z}) + \tan\alpha] - h\int_0^1 s'(z)\,k[K(k)-E(k)]\,dz'\right\} \tag{2.4-3}$$

The complete elliptic integral K(k) has a logarithmic singularity at k = 1, ($\bar{z} = z'$). This causes no difficulty in equation (2.4-2) since $\lim_{\bar{z}\to z'}(\bar{z}-z')^2K(k) = 0$, but the integral in equation (2.4-3) is improper. A logarithmic singularity is removable from the integrand of an integral by a change in variable and this technique will be used later.

Muskhelishvili[26] has shown how to reduce an equation of the type of equation (2.4-1) to a Fredholm equation of the second kind whose solution is known. Using Muskhelishvili's procedure the term g(z'-z') = g(0) is added to and subtracted from the kernal $g(\bar{z}-z')$ in equation (2.4-1) and then this equation becomes

$$\oint_0^1 g(0)\frac{\gamma(z')}{(\bar{z}-z')}\,dz' + \int_0^1[g(\bar{z}-z')-g(0)]\frac{\gamma(z')}{(\bar{z}-z')}\,dz' = H(\bar{z}) \tag{2.4-4}$$

From equation (2.4-2) it follows that g(0) = -2 and equation (2.4-4) is then

$$\oint_0^1 \frac{\gamma(z')}{(\bar{z}-z')}\,dz' = \frac{1}{2}\int_0^1 [2+g(\bar{z}-z')]\frac{\gamma(z')}{(\bar{z}-z')}\,dz' - \frac{1}{2}H(\bar{z}) \qquad (2.4\text{-}5)$$

The integral on the right hand side is not singular at the point $\bar{z} = z'$ but has the indeterminate form 0/0 which can be shown to be zero. By letting the right hand side of this equation equal f $(\bar{z})$ this equation becomes

$$\oint_0^1 \frac{\gamma(z')}{(\bar{z}-z')}\,dz' = f_0(\bar{z}) \qquad (2.4\text{-}6)$$

This is the well-known Cauchy type singular integral equation[26] and has a unique inverse given by

$$\gamma(\bar{z}) = -\frac{1}{\pi\sqrt{\bar{z}(1-\bar{z})}}\left[\frac{1}{\pi}\oint_0^1 \frac{\sqrt{z'(1-z')}}{(z'-\bar{z})}\,f_0(z')\,dz' + 2\int_0^1 \gamma(z')\,dz'\right] \qquad (2.4\text{-}7)$$

The last term is the total circulation about a section of the duct and is a constant. In this equation there is a singularity at the trailing edge of the cylinder ($\bar{z} = 0$, i.e. $z = a_t$) and at the leading edge ($\bar{z} = 1$). In order to satisfy the Kutta condition, equation (2.2-21), the circulation at the trailing edge, $\gamma(0)$, must be made zero.[27] This is accomplished by picking the total circulation so that this singularity is removed. For $\bar{z} = 0$, then

$$\frac{1}{\pi}\oint_0^1 \frac{\sqrt{z'(1-z')}}{z'}\,f_0(z')\,dz' = -2\int_0^1 \gamma(z')\,dz' \qquad (2.4\text{-}8)$$

Substituting this result into equation (2.4-7) removes the singularity at the trailing edge and the circulation distribution is in such a form that the Kutta condition is satisfied.

$$\gamma(\bar{z}) = \frac{1}{\pi^2}\sqrt{\frac{\bar{z}}{1-\bar{z}}}\oint_0^1 \frac{f_o(z')}{(z'-\bar{z})}\sqrt{\frac{(1-z')}{z'}}\,dz' \tag{2.4-9}$$

Substituting in for $f_o(z')$ and interchanging* the order of integration, a Fredholm equation of the second kind is obtained for the circulation $\gamma(\bar{z})$.

$$\gamma(\bar{z}) = -\frac{1}{2\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\int_0^1 \sqrt{\frac{(1-z')}{z'}}\;\frac{H(z')}{(z'-\bar{z})}\,dz'$$

$$+\frac{1}{2\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\int_0^1\left[\oint_0^1\sqrt{\frac{1-z''}{z''}}\;\frac{[2+g(z''-z')]}{(z''-\bar{z})(z''-z')}\,dz''\right]\gamma(z')\,dz' \tag{2.4-10}$$

where

$$g(z''-z') = k\left\{4h^2(z''-z')^2[K(k)-E(k)] - 2E(k)\right\}$$

$$k^2 = \frac{1}{h^2(z''-z')^2+1} \tag{2.4-11}$$

This equation is still not in a form which can easily be solved since $\gamma(\bar{z})$ has a singularity at $\bar{z} = 1$, i.e. the leading edge. To remove this singularity a new dependent variable is defined as $\gamma^*(z) = \sqrt{1-\bar{z}}\;\gamma(\bar{z})$, thus at $\bar{z} = 1$ this new dependent variable is zero. Equation (2.4-10) can then be rewritten as follows

$$\gamma^*(\bar{z}) = \sqrt{1-\bar{z}}\;\gamma(\bar{z}) = f(\bar{z}) + \int_0^1 K_1(\bar{z},z')\frac{\gamma^*(z')}{\sqrt{1-z'}}\,dz' \tag{2.4-12}$$

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* Interchange of the order of integration of two Cauchy principal value integrals would normally result in a residue. The residue is zero here because of the form of the integrand.

where

$$f(\bar{z}) = -\frac{1}{2\pi^2}\sqrt{\bar{z}}\oint_0^1 \frac{1}{(z'-\bar{z})}\sqrt{\frac{1-z'}{z'}}\; H(z')dz'$$

$$K(\bar{z},z') = \frac{1}{2\pi^2}\sqrt{\bar{z}}\oint_0^1 \sqrt{\frac{1-z''}{z''}}\,\frac{[2+g(z''-z')]}{(z''-\bar{z})(z''-z')}\,dz''$$

Both $f(\bar{z})$ and $K_1(\bar{z}, z')$ are Cauchy principle value integrals since the integrand is singular. Since it is desired that a solution m thod be obtained for an arbitrary $H(z)$ and since $g(z'' - z')$ is of such a form that $K(\bar{z}, z')$ cannot be obtained by simple quadratures, a change of variable is made and then certain functions are expanded in a Fourier series. Let $\bar{z} = \frac{1}{2}(1 + \cos\theta)$, $z' = \frac{1}{2}(1 + \cos\theta')$ and $z'' = \frac{1}{2}(1 + \cos\theta'')$, then equation (2.4-12) becomes

$$\gamma^*(\theta) = \sin\tfrac{1}{2}\theta\,\gamma(\theta) = f(\theta) + \int_0^\pi K(\theta,\theta')\,\gamma^*(\theta')d\theta' \tag{2.4-13}$$

where

$$f(\theta) = -\frac{1}{2\pi^2}\cos\tfrac{1}{2}\theta\oint_0^\pi \frac{(1-\cos\theta')}{(\cos\theta'-\cos\theta)}\,H(\theta')d\theta' \tag{2.4-14}$$

$$K(\theta,\theta') = \frac{1}{\pi^2}\cos\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\oint_0^\pi \frac{(1-\cos\theta'')}{(\cos\theta''-\cos\theta)}\left[\frac{2+g(\cos\theta''-\cos\theta')}{(\cos\theta''-\cos\theta')}\right]d\theta'' \tag{2.4-15}$$

To find the Cauchy principal value of the integral for $f(\theta)$, the function $H(\theta')$ will be obtained in a different form. From equation (2.4-14) $H(\theta')$ is obtained as

$$H(\theta') = 4(1-w_{xd})\left\{\pi[C_1'(\theta') + \tan\alpha] - \frac{h}{2}\int_0^\pi s'(\theta'')k[K(k) - E(k)]\sin\theta''d\theta''\right\} \tag{2.4-16}$$

where

$$\ell^2 = \frac{4}{k^2(\cos\theta' - \cos\theta'')^2 + 4}$$

The thickness distribution and the mean line shape are now expanded in a Fourier sine series in θ'. Because of the shape of the airfoil section it would be expected that such a series would converge very rapidly. The use of such a Fourier series implies tha the required slopes can be obtained by a term by term differentiation of the series. For this to be possible the mean line and thickness distributions must satisfy additional restrictions than would normally be necessary for their expansion in a Fourier series. Specifically for the Fourier series of a function to be differentiated term by term[28] the function must be everywhere continuous and possess a derivation which satisfies the Dirichlet conditions. For practical sections this causes no restriction on the shape, even if, the slopes are infinite at the ends since the requirement in this case is that the integral of the slopes be absolutely convergent. Expanding the thickness and camber distribution in a Fourier sine series in θ, the following are obtained.

$$C_l(\theta) = \sum_{m=1}^{\infty} C_m \sin m\theta \qquad (2.4\text{-}17)$$

$$C_m = \frac{2}{\pi}\int_0^{\pi} C_l(\theta)\sin m\theta\, d\theta$$

$$S(\theta) = \sum_{m=1}^{\infty} S_m \sin m\theta \qquad (2.4\text{-}18)$$

$$S_m = \frac{2}{\pi}\int_0^{\pi} S(\theta)\sin m\theta\, d\theta$$

The slopes are obtained by differentiation and are

$$C_1'(\theta) = \sum_{m=1}^{\infty} c_m \, m \cos m\theta$$

$$S'(\theta) = \sum_{m=1}^{\infty} s_m \, m \cos m\theta$$

(2.4-19)

Introducing these expressions for the slopes into equation (2.4-16) and interchanging[28] the order of integration and summation, H(θ) is ob :ined in the following form.

$$H(\theta') = (1 - w_{XJ})\left\{4\pi\left[\tan\alpha + \sum_{m=1}^{\infty} c_m m \cos m\theta\right] - 2h\sum_{m=1}^{\infty} s_m m \int_0^{\pi} k\left[K(k) - E(k)\right]\cos m\theta'' \sin\theta'' d\theta''\right.$$

(2.4-20)

The elliptic integral of the first kind K(k) has a logarithmic singularity at $k = 1$, i.e. when $\cos\theta'' = \cos\theta'$. This singularity can be removed from the integral by considering the change in variable $(\cos\theta' - \cos\theta'') = t^3$, then

$$\int_0^{\pi} k\left[K(k) - E(k)\right]\sin m\theta'' \sin\theta'' d\theta'' = 3\int_{-\sqrt[3]{(1-\cos\theta')}}^{\sqrt[3]{(1+\cos\theta')}} t^2 k\left[K(k) - E(k)\right]\cos m\left\{\arccos\left[\cos\theta' - t^3\right]\right\} dt$$

$$= 3\,G(\theta', m)$$

(2.4-21)

where

$$k^2 = \frac{4}{ht^6 + 4}$$

This removes the singularity and the integral can be evaluated numerically without difficulty. To complete the

solution, the function $G(\theta', m)$ is expanded in a Fourier cosine series in θ', i.e.

$$G(\theta', m) = \int_{-\sqrt[3]{(1-\cos\theta')}}^{\sqrt[3]{(1+\cos\theta')}} t^2 k[K(k) - E(k)] \cos m \left\{ \arccos[\cos\theta' - t^3] \right\} dt = \sum_{p=0}^{\infty} a_p(m) \cos p\theta' \tag{2.4-22}$$

where

$$a_0(m) = \frac{1}{\pi} \int_0^{\pi} G(\theta', m)\, d\theta'$$

$$a_p(m) = \frac{2}{\pi} \int_0^{\pi} G(\theta', m) \sin \quad \theta' d\theta', \; (p = 1, 2, 3 \cdots \infty) \tag{2.4-23}$$

Using this representation of $G(\theta', m)$ and substituting equation (2.4-16) into (2.4-14), the integral for $f(\theta)$ can be evaluated

$$f(\theta) = -\frac{2}{\pi}(1 - w_{xd}) \left\{ \tan\alpha \cos\frac{1}{2}\theta \oint_0^{\pi} \frac{(1-\cos\theta')}{(\cos\theta' - \cos\theta)} d\theta' + \cos\frac{1}{2}\theta \sum_{m=1}^{\infty} C_m m \oint_0^{\pi} \frac{(1-\cos\theta')}{(\cos\theta' - \cos\theta)} \cos m\theta\, d\theta' \right\}$$

$$+ \frac{3h}{\pi^2}(1 - w_{xd}) \cos\frac{1}{2}\theta \sum_{m=1}^{\infty} S_m m \left[\sum_{p=0}^{\infty} a_p(m) \oint_0^{\pi} \frac{(1-\cos\theta')}{(\cos\theta' - \cos\theta)} \cos p\theta' \right] \tag{2.4-24}$$

The integrals occurring in this equation are of the Glauert type and their evaluation is given in Reference [27].

$$\oint_0^{\pi} \frac{(1-\cos\theta')}{(\cos\theta' - \cos\theta)} \cos m\theta' d\theta' = \oint_0^{\pi} \frac{\cos m\theta'}{(\cos\theta' - \cos\theta} d\theta' - \int_0^{\pi} \frac{\cos\theta' \cos m\theta'}{(\cos\theta' - \cos\theta)} d\theta'$$

$$= \pi \frac{\sin m\theta}{\sin\theta} - \oint_0^{\pi} \left[\cos m\theta' + \frac{\cos\theta \cos m\theta'}{(\cos\theta' - \cos\theta)} \right] d\theta'$$

$$= \pi (1 - \cos\theta) \frac{\sin m\theta}{\sin\theta} \tag{2.4-25}$$

Also

$$\int_0^{\pi} \frac{(1-\cos\theta')}{(\cos\theta'-\cos\theta)}\,d\theta' = -\pi$$

Using these values for the integrals, f(θ) follows as

$$f(\theta) = 2(1-w_{xd})\left\{\tan\alpha\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\sum_{m=1}^{\infty}c_m m\sin m\theta\right\}$$

$$+\frac{3h}{\pi}(1-w_{xd})\left\{-\cos\tfrac{1}{2}\theta\sum_{m=1}^{\infty}s_m m\,a_0(m) + \sin\tfrac{1}{2}\theta\sum_{m=1}^{\infty}s_m m\left[\sum_{p=1}^{\infty}a_p(m)\sin p\theta\right]\right\}$$

$$=(1-w_{xd})\left\{\left[2\tan\alpha + \sum_{m=1}^{\infty}s_m F_m\right]\cos\tfrac{1}{2}\theta + \left[-2\sum_{m=1}^{\infty}c_m m\sin m\theta + \sum_{m=1}^{\infty}s_m B_m(\theta)\right]\sin\tfrac{1}{2}\theta\right\}$$

(2.4-26)

where

$$F_m = -\frac{3}{\pi}\,m\,a_0(m)$$

$$B_m(\theta) = \frac{3h}{\pi}\,m\left[\sum_{p=1}^{\infty}a_p(m)\sin p\,\theta\right] \qquad (2.4\text{-}27)$$

The coefficients F_m and $B_m(\theta)$ are independent of the section shape, however, they are dependent on the chord-diameter ratio of the duct, and can be tabulated. Now consider the equation for the kernal K(θ,θ'), equation (2.4-15). Attempts at evaluating this improper integral by simple quadratures have been unsuccessful so K(θ,θ') will be obtained in a form which can be solved by numerical methods. In the following method the term

$$\left[\frac{2+g(\cos\theta''-\cos\theta')}{\cos\theta''-\cos\theta'}\right]$$

is expanded in a Fourier cosine series in θ'' in the range $0 \leq \theta'' \leq \pi$. This function satisfies the Dirichlet conditions[28] so can be expanded in such a series. Furthermore, the function is continuous for $0 \leq \theta'' \leq \pi$ but the first derivative is discontinuous at $\theta'' = \theta'$. Expanding this term in an even series, it becomes

$$\left[\frac{2 + g(\cos\theta'' - \cos\theta')}{(\cos\theta'' - \cos\theta')}\right] = \sum_{n=0}^{\infty} b_n(\theta')\cos n\theta'' \tag{2.4-28}$$

where

$$b_0(\theta') = \frac{1}{\pi}\int_0^{\pi}\left[\frac{2 + g(\cos\theta'' - \cos\theta')}{\cos\theta'' - \cos\theta'}\right] d\theta''$$

$$b_n(\theta') = \frac{2}{\pi}\int_0^{\pi}\left[\frac{2 + g(\cos\theta'' - \cos\theta')}{\cos\theta'' - \cos\theta'}\right]\cos n\theta''\, d\theta'' \tag{2.4-29}$$

The integral for the coefficients must be evaluated numerically but some difficulty arises at the point $\theta'' = \theta'$ where the integrand has an indeterminate form. To determine the value at this point l'Hospital's rule is applied to the integrand.

$$\lim_{\theta'' \to \theta'}\left[\frac{2 + g(\cos\theta'' - \cos\theta')}{\cos\theta'' - \cos\theta'}\right]\cos n\theta'' = -\frac{g_{\theta''}(0)\cos n\theta'}{\sin\theta'} = 0 \tag{2.4-30}$$

Since

$$g_{\theta''}(0) = \lim_{\theta'' \to \theta'} g_{\theta''}(\cos\theta'' - \cos\theta') = \lim_{\theta'' \to \theta'}\left[-\sin\theta''(\cos\theta'' - \cos\theta')\frac{k}{16}\left\{\left[\frac{(\cos\theta'' - \cos\theta')^3 k^2 - 1}{16}\right]\right.\right.$$

$$\left.\left.\cdot\left[K(k) - E(k)\right] + \frac{k^2}{2}K(k)\right\}\right] = 0$$

At the point $\theta'' = \theta'$, therefore, the integrand is zero, however, because of the form of equation (2.4-30), it must be shown that when $\theta' = 0$ or π equation (2.4-30) is still valid. Following the same procedure as above for $\theta' = 0$, then

$$\lim_{\theta''\to 0}\left[\frac{2+g(\cos\theta''-1)}{\cos\theta''-1}\right]\cos n\theta'' = \lim_{\theta''\to 0}\left[\frac{g_{\theta''}(\cos\theta''-1)\cos\theta'' - n\sin n\theta''[2+g(\cos\theta''-1)]}{-\sin\theta''}\right]$$

$$= \lim_{\theta''\to 0}\left[(\cos\theta''-1)\frac{k}{16}\left\{\left[\frac{(\cos\theta''-1)^2 k^2-1}{16}\right]\left[K(k)-E(k)\right]+\frac{k^2}{2}K(k)\right\}+\frac{n\sin n\theta''}{\sin\theta''}\left[2+g(\cos\theta''-1)\right]\right]$$

$$= \lim_{\theta''\to 0}\left[\frac{(n\sin n\theta'')g_{\theta''}(\cos\theta''-1)+n^2\cos n\theta''[2+g(\cos\theta''-1)]}{\cos\theta''}\right] = 0$$

From this last result and equation (2.4-30) it can be concluded that the integrand of equation (2.4-28) has the value of zero for $\theta'' = \theta'$, $0 \le \theta' \le \pi$. Substituting equation 2.4-28) into the kernal, equation (2.4-15), the integration can be performed.

$$K(\theta,\theta') = \frac{1}{\pi^2}\cos\frac{1}{2}\theta'\cos\frac{1}{2}\theta\int_0^{\pi}\frac{(1-\cos\theta'')}{(\cos\theta''-\cos\theta)}\left[\sum_{n=0}^{\infty}b_n(\theta')\cos n\theta''\right]d\theta''$$

$$= \frac{1}{\pi^2}\cos\frac{1}{2}\theta'\cos\frac{1}{2}\theta\sum_{n=0}^{\infty}b_n(\theta')\left[\int_0^{\pi}\frac{(1-\cos\theta'')}{(\cos\theta''-\cos\theta)}\cos n\theta''\,d\theta''\right]$$

$$= \frac{1}{\pi}\cos\frac{1}{2}\theta'\left[-(\cos\frac{1}{2}\theta)\,b_0(\theta') + (\sin\frac{1}{2}\theta)\sum_{n=1}^{\infty}b_n(\theta')\sin n\theta\right] \qquad (2.4\text{-}31)$$

$K(\theta, \theta')$ is a known function involving only the chord-diameter ratio (h) as a parameter so can be tabulated. Substituting for $f(\theta)$, equation (2.4-26), and the kernal $K(\theta, \theta')$, equation (2.4-31), into equation (2.4-13) the Fredholm equation of the second kind for the circulation distribution is obtained in a form which can be solved by known methods.

$$\gamma^*(\theta) = (1 - W_{Xd})\left\{\left[2\tan\alpha + \sum_{m=1} s_m F_m\right]\cos\tfrac{1}{2}\theta + \left[-2\sum_{m=1}^{\infty} c_m m \sin m\theta + \sum_{m=1}^{\infty} s_m B_m(\theta)\right]\sin\tfrac{1}{2}\theta\right\}$$

$$+\frac{1}{\pi}\int_0^{\pi}\left\{\cos\tfrac{1}{2}\theta'\left[-(\cos\tfrac{1}{2}\theta)\, b_0(\theta') + (\sin\tfrac{1}{2}\theta)\sum_{n=1}^{\infty} b_n(\theta')\sin n\theta\right]\right\}\gamma^*(\theta')\,d\theta' \qquad (2.4\text{-}32)$$

II.5 Ideal Angle of Attack

In the integral equation (2.4-10) for the circulation distribution a singularity exists at the leading edge of the ring which made it necessary to redefine the circulation distribution. This singularity cannot exist if the ring section is designed so that a stagnation point occurs at the leading edge. The angle of attack at which the section is operating when a stagnation point occurs at the leading edge is known as the ideal angle of attack (α_{id}). Since an axi-symmetric annular airfoil has been assumed, it is obvious that an ideal angle of attack cannot be defined if the radial velocities on the duct surface are functions of angle. Consequently, in the presence of a propeller an ideal angle of attack cannot be defined which applies to every section.

To remove the singularity occurring at the leading edge, i.e., $\bar{z} = 1$, in equation (2.4-9), this equation is rewritten as follows:

$$\gamma(\bar{z}) = \frac{1}{\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}} \oint_0^1 \frac{1}{(z'-\bar{z})}\sqrt{\frac{(1-z')}{z'}}\left[f_1(z') - 2(1-w_{xd})\,\pi\tan\alpha\right]dz'$$

$$= \frac{1}{\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\left[2(1-w_{xd})\pi^2\tan\alpha + \oint_0^1 \frac{f_1(z')}{(z'-\bar{z})}\sqrt{\frac{1-z'}{z'}}\,dz'\right] \qquad (2.5\text{-}1)$$

where

$$f_1(z') = f_0(z') + 2(1-w_{xd})\,\pi\tan\alpha$$

Following the arguments for satisfying the Kutta condition, a stagnation point will occur at the leading edge of the duct if the circulation at that point is zero. The circulation will

be zero at the leading edge if the angle α is chosen so that the terms in the brackets of the previous equation cancel out the singularity of the term $\frac{1}{\sqrt{1-\bar{z}}}$ at $\bar{z} = 1$. The ideal angle of attack is therefore, defined as follows

$$\tan \alpha_{id} = -\frac{1}{2\pi^2(1-w_{xd})}\oint_0^1 \frac{f_1(z')}{z'-1}\sqrt{\frac{1-z'}{z'}}\, dz'$$

$$= \frac{1}{2\pi^2(1-w_{xd})}\oint_0^1 \frac{f_1(z')}{\sqrt{(1-z')z'}}\, dz' \tag{2.5-2}$$

The term $f_1(z')$ occurring in this equation is a function of the circulation distribution $\gamma(\bar{z})$ and this distribution must correspond to that occurring at the ideal angle of attack. To determine the so called ideal lift coefficient, $\gamma_{id}(\bar{z})$ equation (2.5-2) is substituted into equation (2.5-1). The solution of this integral equation will give the circulation distribution for the section operating at its ideal angle of attack.

$$\gamma_{id}(\bar{z}) = \frac{1}{\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\left[\int_0^1 \frac{f_1(z')}{\sqrt{(1-z')z'}}\, dz' + \oint_0^1 \frac{f_1(z')}{(z'-\bar{z})}\sqrt{\frac{1-z'}{z'}}\, dz'\right]$$

$$= \frac{1}{\pi^2}\sqrt{\bar{z}(1-\bar{z})}\oint_0^1 \frac{f_1(z')\, dz'}{(z'-\bar{z})\sqrt{z'(1-z')}} \tag{2.5-3}$$

From the form of this equation it can be seen that the singularity is now removed from the leading edge. Proceeding as in the last section it can be shown that this equation can be reduced to a Fredholm equation of the second kind for

the circulation distribution. The following form is obtained for this equation where the coefficients are the same as given in the previous section.

$$\gamma_{id}(\theta) = (1 - w_{xd})\left[-2\sum_{m=1}^{\infty} c_m m \sin m\theta + \sum_{m=1}^{\infty} s_m B_m(\theta)\right]$$

$$+ \frac{1}{2\pi}\int_0^{\pi}\left[\sin\theta' \sum_{n=1}^{\infty} b_n(\theta') \sin n\theta\right]\gamma_{id}(\theta')\,d\theta' \qquad (2.5\text{-}3)$$

By substituting into equation (2.5-2) it follows that the ideal angle of attack can be written as:

$$\tan\alpha_{id} = \frac{1}{4\pi(1 - w_{xd})}\int_0^{\pi} b_o(\theta')\gamma_{id}(\theta')\sin\theta'\,d\theta' - \frac{1}{2}\sum_{m=1}^{\infty} s_m F_m \qquad (2.5\text{-}4)$$

II.6 Solution of the Fredholm Equation of the Second Kind for the Circulation Distribution.

Both equations (2.4-32) and (2.5-3) can be solved by the same procedure. The general form for either equation can be written as follows

$$\gamma(\theta) = (1 - W_{xd})\left\{B_o(\theta) + B(\theta)\left[-2\sum_{m=1}^{\infty} c_m\, m \sin m\theta + \sum_{m=1}^{\infty} s_m B_m(\theta)\right]\right\}$$

$$+ c(\theta)\int_0^{\pi}\left\{D(\theta')\left[D_o(\theta)\, b_o(\theta') + \sum_{n=1}^{\infty} b_n(\theta') \sin n\theta\right]\right\}\gamma(\theta')\, d\theta' \tag{2.6-1}$$

In the general case:

$$\gamma(\theta) = \gamma^*(\theta)$$

$$B_o(\theta) = [2 \tan\alpha + \sum_{m=1}^{\infty} S_m F_m] \cos \tfrac{1}{2}\theta$$

$$B(\theta) = \sin \tfrac{1}{2}\theta$$

$$C(\theta) = \tfrac{1}{\pi} \sin \tfrac{1}{2}\theta$$

$$D(\theta') = \cos \tfrac{1}{2}\theta'$$

$$D_o(\theta) = -\cot \tfrac{1}{2}\theta$$

and for the ideal circulation distribution

$$\gamma(\theta) = \gamma_{id}(\theta)$$

$$B_o(\theta) = 0$$

$$B(\theta) = 1$$

$$C(\theta) = \frac{1}{2\pi}$$

$$D(\theta') = \sin\theta'$$

$$D_o(\theta) = 0$$

The Fourier coefficients are the same as defined in Section II.4. The integral equation (2.6-1) can be solved by the method of successive approximation or the Fredholm solution method.[29] In addition, since the kernal in this equation is a degenerate (or product) kernal, the special method appropriate to this type of kernal can be used. Which method is best depends on the convergence of the Fourier series representing the kernal. If the series converges rapidly the method appropriate to product kernal is probably best. Following is an outline of this method, necessary proof of the convergence is given in Reference [29].

To apply this method the order of integration and summation are interchanged in equation (2.6-1) and then the equation can be written as

$$\gamma(\theta) = f(\theta) + c(\theta)\Big[D_0(\theta)\int_0^{\pi} b_0(\theta')D(\theta')\gamma(\theta')d\theta' + \sin\theta\int_0^{\pi} b_1(\theta')D(\theta')\gamma(\theta')d\theta'$$

$$+ \sin 2\theta\int_0^{\pi} b_2(\theta')D(\theta')\gamma(\theta')d\theta' + \cdots + \sin n\theta\int_0^{\pi} b_n(\theta')D(\theta')\gamma(\theta')d\theta' \qquad (2.6\text{-}2)$$

where

$$f(\theta) = (1 - W_{xd})\Big\{B_0(\theta) + B(\theta)\Big[-2\sum_{m=1}^{\infty} c_m\, m \sin m\theta + \sum_{m=1}^{\infty} s_m B_m(\theta)\Big]\Big\}$$

For convenience let

$$A_i = \int_0^{\pi} b_n(\theta')D(\theta')\gamma(\theta')d\theta' \;, \quad (i = 0, 1, 2, \cdots n)$$

and then the above equation can be rewritten as:

$$\gamma(\theta) = f(\theta) + c(\theta)D_0(\theta)A_0 + c(\theta)A_1\sin\theta + \cdots + c(\theta)A_n\sin n\theta \qquad (2.6\text{-}3)$$

By substituting this equation into equation (2.6-2) the A_j's are obtained.

$$A_0 = \int_0^{\pi} D(\theta')b_0(\theta')\left[f(\theta') + C(\theta')D_0(\theta')A_0 + C(\theta')A_1\sin\theta' + \cdots C(\theta')A_n\sin n\theta'\right]d\theta'$$

$$A_1 = \int_0^{\pi} D(\theta')b_1(\theta')\left[f(\theta') + C(\theta')D_0(\theta')A_0 + C(\theta')A_1\sin\theta' + \cdots C(\theta')A_n\sin n\theta'\right]d\theta'$$

....

....

$$A_n = \int_0^{\pi} D(\theta')b_n(\theta')\left[f(\theta') + C(\theta')D_0(\theta')A_0 + C(\theta')A_1\sin\theta' + \cdots C(\theta')A_n\sin n\theta'\right]d\theta' \tag{2.6-4}$$

In deriving the equacion for A_0, it was necessary to divide both sides of the equation by $D_0(\theta)$ and if $D_0(\theta)$ is zero this is not valid. For this case then, i.e. ideal circulation distribution, A_0 must be taken as zero and the above set of (n + 1) equations reduces to a set of (n) equations. For convenience the following notation is introduced into the preceding equation.

$$c_{ij} = \int_0^{\pi} D(\theta')b_i(\theta')C(\theta')D_j(\theta')d\theta' \qquad , (i, j = 0, 1, 2, \cdots n)$$

$$= \frac{1}{2\pi}\int_0^{\pi} b_i D_j(\theta')\sin\theta' d\theta' \tag{2.6-5}$$

where

$$D_0(\theta') = -\cot\tfrac{1}{2}\theta'$$

$$D_j(\theta') = \sin j\theta', \quad (j = 1, 2, \cdots n)$$

and

$$d_i = \int_0^{\pi} D(\theta')b_i(\theta')f(\theta')d\theta'$$

$$= (1 - W_{\bar{x}d}) \left[\left\{ 2\tan\alpha + \sum_{m=1}^{\infty} s_m F_m \right\} f_i - N \sum_{m=1}^{\infty} c_m d_{im} + \frac{N}{2} \sum_{m=1}^{\infty} s_m f_{im} \right] \tag{2.6-6}$$

where

$$f_i = \int_0^{\pi} b_i(\theta') \cos^2 \frac{1}{2} \theta' d\theta'$$

$$d_{im} = m \int_0^{\pi} b_i(\theta') \sin\theta' \sin m\theta' d\theta'$$

$$f_{im} = \int_0^{\pi} b_i(\theta') B_m(\theta') \sin\theta' d\theta'$$

and

N = 1 for the general case

N = 2 for the ideal case

In the foregoing equations the coefficients c_{ij}, f_i, d_{im} and f_{im} are independent of the section shape and can be tabulated for various chord-diameter ratios.

Introducing equation (2.6-5) and equation (2.6-6) into equation (2.6-4) results in a set of simultaneous equations for the coefficients A_n.

$$\begin{aligned}
A_o(1 - c_{oo}) - A_1 c_{o1} - A_2 c_{o2} - \cdots - A_n c_{on} &= d_o \\
- A_o c_{1o} + A_1(1 - c_{11}) - A_2 c_{12} - \cdots - A_n c_{1n} &= d_1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
- A_o c_{no} - A_1 c_{n1} - A_2 c_{n2} - \cdots + A_n(1 - c_{nn}) &= d_n
\end{aligned} \tag{2.6-7}$$

This system of equations represents an algebraic set of simultaneous equations for the unknown A_n's. The existance of a unique solution depends on the determinent of the coefficients of A_n on the left hand side being different from zero (Cramer's rule).[28] The number of simultaneous equations depends on the number of terms needed in the Fourier series so it satisfactorily approximates the kernal. Fortunately this can be determined once and for all since the kernal, $K(\theta, \theta')$, is independent of section shape.

II.7 Pressure Distribution and Forces on the Annular Airfoil

The velocity field of the ducted propeller is found by summing the free-stream velocity, the velocity induced by the duct including the trailing vortex system, and the velocity induced by the propeller and hub. Since the flow field has been assumed to be irrotational, steady and incompressible and the body forces have been neglected, Bernoulli's equation can be written as follows:

$$\frac{p(x,\varphi,z)-p_o}{\frac{1}{2}\rho V_s^2(1-w_x)^2} = -\left(\left[-1+\frac{w_a(x,\varphi,z)}{V_s(1-w_x)}\right]^2 - 1 + \left[\frac{w_r(x,\varphi,z)}{V_s(1-w_x)}\right]^2 + \left[\frac{w_t(x,\varphi,z)}{V_s(1-w_x)}\right]^2\right) \tag{2.7-1}$$

The axial velocity $w_a(x, \varphi, z)$, radial velocity $w_r(x, \varphi, z)$ and tangential velocity $w_t(x, \varphi, z)$ are the total velocities induced by the various singularities in the flow and are commonly called perturbation velocities. The pressure p_o is the pressure infinitely far ahead of the propeller while $p(x, \varphi, z)$ is the local pressure. This pressure distribution has been nondimensionalized by the ship velocity times the wake $(1 - w_x)$ which is the local free-stream velocity.

If this equation is linearized by the same method as used for the linearized boundary conditions, the squared terms will be neglected and the linearized prressure distribution is then

$$\frac{p(x,\varphi,z)-p_o}{\frac{1}{2}\rho V_s^2(1-w_x)^2} = \frac{2\,w_a(x,\varphi,z)}{V_s^2(1-w_x)^2} \tag{2.7-2}$$

The pressure on the annular airfoil itself is found by substituting the duct radius x_d for x in the preceding equation.

$$\frac{p(x_d, \varphi, z) - p_0}{\frac{1}{2}\rho V_s^2 (1 - w_{xd})^2} = \frac{2\, w_a(x_d, \varphi, z)}{V_s^2 (1 - w_{xd})^2} \tag{2.7-3}$$

The velocity $w_a(x, \varphi, z)$ is the axial velocity induced by all the singularities in the flow. Also, it should be noted that since the velocity induced by the vortex cylinder is discontinuous across the cylinder, equation (A-15), the pressure changes from the inside to the outside of the duct.

In the problem of a ducted propeller in an inviscid fluid, the only net force on the duct itself is the so-called induced drag or force in the axial dirextion. There is a radial force on each section which contributes to hoop stress but because of symmetry of the flow this net force is zero. Since the net lateral force is zero, there is no moment on the duct, however a moment on each individual section could be defined.

The force F on any section of the duct is given by the Kutta-Joukowski law[27] which can be expressed as

$$F = \rho V \Gamma \tag{2.7-4}$$

The velocity V is the velocity by the annular airfoil section perpendicular to the direction of the force and Γ is the total circulation about each section. The velocity V

does not include the self induced velocities so does not contain the velocities induced by the vortex and source rings. Assuming that both the velocity and circulation $\gamma(\varphi, z)$ are nondimensionalized by the ship speed and the axial coordinate $\bar{z}$ by the chord, the lift at each section is

$$dL = \rho a V_s^2 \int_0^1 \gamma(\varphi, \bar{z}) \left[-(1 - w_{x_d}) + \frac{v_a}{V_s}(x_d, \varphi, \bar{z})_{\varphi + \frac{\partial\gamma}{\partial\varphi}} \right] d\bar{z} \tag{2.7-5}$$

The induced velocity $v\cdot(x_d, \varphi, z)_{\varphi + \frac{\partial\gamma}{\partial\varphi}}$ is the axial velocity induced on the cylinder of radius x_d by the propeller, hub and duct trailing vortex system. If the total lift on the whole ring is taken to be some arbitrary direction with respect to a propeller blade, normally in the vertical direction and positive upward, the contribution to the lift of any arbitrary section of the ring would be

$$dL \cos\varphi = (\cos\varphi)\rho a V_s^2 \int_0^1 \gamma(\varphi, \bar{z}) \left[-(1 - w_{x_d}) + \frac{v_a}{V_s}(x_d, \varphi, \bar{z})_{\varphi + \frac{\partial\gamma}{\partial\varphi}} \right] d\bar{z} \tag{2.7-6}$$

The angle φ is measured from the arbitrarily taken direction which is attached to one of the propeller blades, i.e. rotating coordinate system. If this equation is integrated completely around the circumference of the ring, the net lift will be zero since the flow is axisymmetric. If the annular airfoil is at an angle of attack to the flow, the flow is no longer axisymmetric and there is a net lift force.

The thrust force on the duct, which is equal and opposite to the induced drag, is also given by the Kutta-Joukowski law, equation (2.7-4), but in this case the

velocity at the duct is the radial velocity, again not including the self induced velocities. Denoting the duct thrust by T_d , the following equation results for the thrust force on each section.

$$dT_d = \rho a V_s^2 \int_0^1 \gamma(\bar{z}, \varphi) \left[\frac{w_r}{V_s}(x_d, \varphi, \bar{z})_{p+h} + \frac{w_r}{V_s}(x_d, \varphi, \bar{z})_{\frac{\partial \gamma}{\partial \varphi}} \right] d\bar{z} \tag{2.7-7}$$

and the total thrust is given by integrating this equation around the ring.

$$T_d = \rho a V_s^2 \int_0^1 \int_0^{2\pi} \gamma(\bar{z}, \varphi) \left[\frac{w_r}{V_s}(x_d, \varphi, \bar{z})_{p+h} + \frac{w_r}{V_s}(x_d, \varphi, \bar{z})_{\frac{\partial \gamma}{\partial \varphi}} \right] x_d R_p \, d\varphi \, d\bar{z} \tag{2.7-8}$$

This equation can be put in the form of the thrust coefficient used in propeller design by dividing by $\frac{\rho}{2} R_p^2 \pi V_s^2$.

$$\left(C_{T_{s,i}}\right)_d = \frac{T_d}{\frac{\rho}{2} R_p^2 \pi V_s^2} = \frac{4h x_d^2}{\pi} \int_0^1 \int_0^{2\pi} \gamma(\bar{z}, \varphi) \left[\frac{w_r}{V_s}(x_d, \varphi, \bar{z})_{p+h} + \frac{w_r}{V_s}(x_d, \varphi, \bar{z})_{\frac{\partial \gamma}{\partial \varphi}} \right] d\varphi \, d\bar{z} \tag{2.7-9}$$

The subscript "i" in this equation means inviscid fluid.

II.8 Linearized Theory of the Annular Airfoil at an Angle of Incidence.

In addition to the assumptions made previously concerning the annular airfoil, it will now be assumed that the free-stream velocity is a constant but at an angle α_r to the ring. The angle will be assumed to be small enough so that $\sin\alpha_r = \tan\alpha_r = \alpha_r$ and $\cos\alpha_r = 1$. As previously, a cylindrical coordinate system $(x, \varphi, \bar{z})$ will be used with the zero axial coordinate $(\bar{z})$ zero at the trailing edge and (R_d) the reference diameter will be located at the propeller centerline.

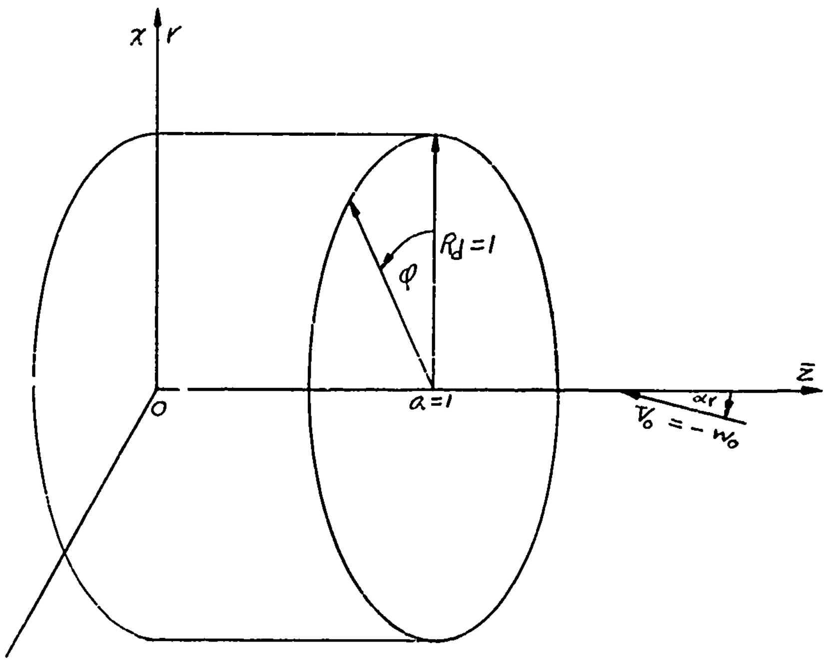

Figure 5. Free stream velocity at an angle to the duct

Applying the perturbation theory as in Section II.2, and assuming the angle of incidence α_r to be small, it can be

shown that the linearized kinematic boundary condition is

$$\frac{w_r}{V_0}(R_d \pm 0, \varphi, \bar{z}) + w_0 \alpha_r \cos\varphi = -w_0[c'(\xi) \pm s'(\xi)], \ (0 \leq \bar{z} \leq 1) \quad (2.8\text{-}1)$$

In addition to this boundary condition, the Kutta condition must be satisfied at the trailing edge of the ring.

$$\varphi_{\bar{z}}(R_d \pm 0, \varphi, 0) = 0 \quad (2.8\text{-}2)$$

One difference betwe n equation (2.8-1) and the boundary condition for the annular airfoil at zero incidence is the addition of a radial velocity term from the free-stream velocity which is dependent on the angular position. This then implies that the radial velocity induced by the annular airfoil must also be a function of the angular position or with reference to the mathematical model the ring vortex strength is a function of angle. As for a three-dimensional wing[27] with a spanwise change in vortex strength, the fact that the vortex strength is a function of angular position leads to a trailing vortex sheet. This vortex sheet is assumed to be cylindrical in shape and to extend to infinity behind the annular ring.

The radial velocity w_r in equation (2.8-1) is mathematically conceived as being a sum of the radial velocity induced on the ring by the ring vortices, ring sources and trailing vortex system. The integral equation for the ring vortex and source strength was derived in Section II.3, equation (2.3-3), and further the source strength was shown to be independent of the angular position, equation (2.3-4). The velocity induced

by the trailing vortex system from the duct at an angle of attack follows from the law of Biot-Savart, and is derived in Appendix C, specifically equation (C-12). If this equation is substituted into equation (2.3-3) along with equation (2.3-4) an integral equation for the vortex circulation is obtained. In this section the flow is assumed uniform so $(1 - w_{xd}) = 1$.

$$4h\int_0^1 h(\bar{z}-z')\left[\int_0^{2\pi}\frac{\cos(\varphi-\varphi')\,\gamma(\varphi,z')\,d\varphi'}{[4h^2(\bar{z}-z')^2+4\sin^2\frac{1}{2}(\varphi-\varphi')]^{3/2}}\right]dz'$$

$$+\,h\int_0^1\int_0^{2\pi}\left[\cot\tfrac{1}{2}(\varphi-\varphi')\right]\left[\frac{4h(\bar{z}-z')}{[4h^2(\bar{z}-z')^2+4\sin^2\frac{1}{2}(\varphi-\varphi')]^{1/2}}+1\right]\frac{\partial\gamma}{\partial\varphi'}\,d\varphi'dz'$$

$$=-4\left[\pi\left[c'(\bar{z})+\alpha_r\cos\varphi\right]+h\int_0^1 s'(z')\,k\,[K(k)-E(k)]\,dz'\right]$$

$$=-H(\bar{z})-4\pi\alpha_r\cos\varphi=H_2(\bar{z},\varphi) \qquad (2.8\text{-}3)$$

where

$$k^2=\frac{1}{h^2(\bar{z}-z')^2+1}\quad,\quad 0\le\bar{z}\le 1$$

The term $H(\bar{z})$ is the same as given by equation (2.4-3) and the evaluation of the integral in $H(\bar{z})$, which has a singularity in the integrand, is discussed in Section II.4.

Equation (2.8-4) is a singular integral equation of two variables for the circulation distribution. The inversion

of an integral equation of this type is not in general possible. Weissinger[19] encountered this integral in the problem of the thin annular airfoil at incidence and reduced the two variable integral for the circulation to a one-dimensional integral in terms of the Fourier coefficients for the circulation. Following his procedure the circulation $\gamma(\bar{z},\varphi)$ and the function $H_2(\bar{z},\varphi)$ are expanded in a Fourier cosine series in φ . This involves a restriction on the circulation that it must [illegible] continuous in the angular direction, but from a practical point of view this presents no difficulty.

$$\gamma(z',\varphi') = \sum_{n=0}^{\infty} g_n(z')\cos n\varphi' \tag{2.8-4}$$

$$H_2(\bar{z},\varphi) = \sum_{n=0}^{\infty} u_n(\bar{z})\cos n\varphi \tag{2.8-5}$$

where

$$\begin{aligned} g_0(z') &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(\varphi', z')\,d\varphi' \\ g_n(z') &= \frac{1}{\pi}\int_{-\pi}^{\pi} \gamma(\varphi', z')\cos n\varphi'\,d\varphi' \end{aligned} \tag{2.8-6}$$

$$\begin{aligned} u_0(\bar{z}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_2(\varphi,\bar{z})\,d\varphi \\ u_n(\bar{z}) &= \frac{1}{\pi}\int_{-\pi}^{\pi} H_2(\varphi,\bar{z})\cos n\varphi\,d\varphi \end{aligned} \tag{2.8-7}$$

A cosine series is used since the flow is symmetric about a vertical plane in the direction of the z-axis (See Figure 5) and hence the circulation and radial velocities are even functions of φ .

Substituting equations (2.8-4) and (2.8-5) into equation (2.8-3) and interchanging the order of summation and integration, the following is obtained

$$\sum_{n=0}^{\infty} u_n(\bar{z})\cos n\varphi = \sum_{n=0}^{\infty}\Bigg(2h\int_0^1 g_n(z')\,2h(\bar{z}-z')\Bigg[\int_{-\pi}^{\pi}\frac{\cos(\varphi-\varphi')\cos n\varphi'\,d\varphi'}{[4h^2(\bar{z}-z')^2+4\sin^2\frac{1}{2}(\varphi-\varphi')]^{3/2}}\Bigg]dz'$$

$$-nh\int_0^1 g_n(z')\,2h(\bar{z}-z')\Bigg[\int_{-\pi}^{\pi}\frac{\sin n\varphi'\cot\frac{1}{2}(\varphi-\varphi')\,d\varphi'}{[4h^2(\bar{z}-z')^2+4\sin^2\frac{1}{2}(\varphi-\varphi')]^{3/2}}\Bigg]dz'$$

$$-nh\int_0^1 g_n(z')\Bigg[\oint_{-\pi}^{\pi}\sin n\varphi'\cot\tfrac{1}{2}(\varphi-\varphi')\,d\varphi'\Bigg]dz'\Bigg)$$

(2.8-8)

but, see Reference [27],

$$\oint_{-\pi}^{\pi}\sin n\varphi'\cot\tfrac{1}{2}(\varphi-\varphi')\,d\varphi' = 2\oint_0^{\pi}\frac{\sin n\varphi'\sin\varphi'}{\cos\varphi'-\cos\varphi}\,d\varphi' = -2\pi\cos n\varphi \tag{2.8-9}$$

and making a change of variable, $\theta' = (\frac{\varphi-\varphi'}{2})$ and using the trigonometric identity $\cos n\varphi' = \cos 2n\theta'\cos n\varphi - \sin 2n\theta'\sin n\varphi$

$$\int_{-\pi}^{\pi}\frac{\cos(\varphi-\varphi')\cos n\varphi'\,d\varphi'}{[4h^2(\bar{z}-z')^2+4\sin^2\frac{1}{2}(\varphi-\varphi')]^{3/2}} = -2\int_{\frac{\varphi}{2}+\frac{\pi}{2}}^{\frac{\varphi}{2}-\frac{\pi}{2}}\frac{\cos 2\theta'\cos n\varphi'\,d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}}$$

$$= 2\cos n\varphi\int_{-\pi/2}^{\pi/2}\frac{\cos 2n\theta'\cos 2\theta'\,d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}} - 2\sin n\varphi\int_{-\pi/2}^{\pi/2}\frac{\sin 2n\theta'\cos 2\theta'\,d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}}$$

$$= 4\cos n\varphi\int_0^{\pi/2}\frac{\cos 2n\theta'\cos 2\theta'\,d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}} \tag{2.8-10}$$

The integral involving sin $2n\theta'$ is zero because it is an odd function of θ'. Again introducing the new variable $\theta' = \left(\frac{\varphi - \varphi'}{2}\right)$ and the trigonometric identity $\sin n\varphi' = \sin 2n\theta' \cos n\varphi + \cos 2n\theta' \sin n\varphi$

$$\int_{-\pi}^{\pi} \frac{\sin n\varphi' \cot\frac{1}{2}(\varphi-\varphi')d\varphi'}{[4h^2(\bar{z}-z')^2+4\sin^2(\varphi-\varphi)]^{1/2}} = -4\cos n\varphi \int_0^{\pi/2} \frac{\sin 2n\theta' \cot\theta' d\theta}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{1/2}} \tag{2.8-11}$$

Substituting equation (2.8-9), (2.8-10) and (2.8-11) into equation (2.8-8), the following equation is obtained.

$$\sum_{n=0}^{\infty} u_n(\bar{z})\cos n\varphi = \sum_{n=0}^{\infty} \cos n\varphi \cdot 2h\left(\int_0^1 g_n(z')2h(\bar{z}-z')\left[\int_0^{\pi/2} \frac{4(\cos 2n\theta' \cos 2\theta' + n\sin 2n\theta' \sin 2\theta')d\theta'}{[4h^2(\bar{z}-z') + 4\sin^2\theta']^{3/2}}\right.\right.$$

$$\left.\left. + 8nh^2(\bar{z}-z')^2 \int_0^{\pi/2} \frac{\sin 2n\theta' \cot\theta' d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}}\right] + \pi n \int_0^1 g_n(z')dz'\right) \tag{2.8-12}$$

Since a Fourier series is unique, the coefficients can be equated and an integral equation of one variable is obtained for the Fourier coefficients.

$$u_n(\bar{z}) = 2h\int_0^1 g_n(z')2h(\bar{z}-z')\left[\int_0^{\frac{\pi}{2}} \frac{4(\cos 2n\theta' \cos 2\theta' + n\sin 2n\theta' \sin 2\theta')}{[4h^2(\bar{z}-z')^2 + 4\sin^2\theta']^{3/2}}d\theta'\right.$$

$$\left. + 8nh^2(\bar{z}-z')^2 \int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta' \cot\theta' d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}}\right] + 2\pi nh \int_0^1 g_n(z')dz' \tag{2.8-13}$$

Before proceeding further with the solution of equation (2.8- 3) the Fourier coefficients $u_n(\bar{z})$ will be examined.

Substituting for $H_2(\varphi, \bar{z})$ into (2.8-7) the $u_n(\bar{z})$'s are given as

$$u_o(\bar{z}) = -H(\bar{z}) \tag{2.8-14}$$

$$u_1(\bar{z}) = -4\pi\alpha_r \tag{2.8-15}$$

$$u_n(\bar{z}) = 0 \qquad (n = 2,3\ldots\infty)$$

From these equations it can be concluded that all g_n's for $n = 2, 3 \ldots \infty$ are zero and therefore only g_o and g_1 exist, also that g_o is a f nction of shape only and g_1 of angle of attack. First examine the circulation coefficient g_o, letting $n = 0$ in equation (2.8-13).

$$H(\bar{z}) = -2h\int_0^1 g_o(z')\,2h(\bar{z}-z')\left[\int_0^{\frac{\pi}{2}} \frac{4\cos 2\theta' d\theta'}{[4h^2(\bar{z}-z')^2 + \sin^2\theta']^{3/2}}\right]dz'$$

$$= \int_0^1 \frac{g_o(z')}{(\bar{z}-z')}\,k\left(4h^2(\bar{z}-z')\left[K(k)-E(k)\right] - 2E(k)\right)dz' \tag{2.8-16}$$

This equation is of course exactly the same as (2.4-1), thus it can be concluded that $g_o(\bar{z}) = \gamma(\bar{z})$ where $\gamma(\bar{z})$ is the circulation distribution for the annular airfoil at zero incidence. This equation (2.8-16) is solved in Sections II.4 and II.6. It should be mentioned that on applying the Kutta condition, equation (2.8-2), the equation for the Fourier coefficients g_n, equation (2.8-6), implies $g_o(0) = g_1(0) = 0$.

As has been stated, $g_1(\bar{z})$ is independent of the shape of the airfoil and dependent only on the angle of attack and

chord-diameter ratio h . For n = 1 (2.8-13) becomes

$$-4\pi\alpha_r = 4h\int_0^1 g_1(z')h(\bar{z}-z')\left[\int_0^{\frac{\pi}{2}}\frac{4(\cos^2 2\theta' + \sin^2 2\theta')d\theta'}{[4h^2(\bar{z}-z')^2 + 4\sin^2\theta']^{3/2}}\right.$$

$$\left.+8h^2(\bar{z}-z')^2\int_0^{\frac{\pi}{2}}\frac{2\cos^2\theta'\ d\theta'}{[4h^2(\bar{z}-z')^2+4\sin^2\theta']^{3/2}}\right]dz' + 2\pi h\int_0^1 g_1(z')dz'$$

or

$$-4\pi\alpha_r = 2\int_0^1\frac{g_1(z')}{(\bar{z}-z')}\left(\frac{1}{k^3}\right)\left[(2-k^2)^2E(k) - 4(1-k^2)^2K(k)\right]dz' + 2\pi h\int_0^1 g_1(z')dz'$$

$$= 2\int_0^1\frac{g_1(z')}{(\bar{z}-z')}W_1(\bar{z}-z')dz' + 2\pi h\int_0^1 g_1(z')dz'$$

(2.8-17)

where

$$W_1(\bar{z}-z') = \frac{1}{k^3}\left[(2-k^2)^2E(k) - 4(1-k^2)^2K(k)\right] \qquad (2.8\text{-}18)$$

$$k^2 = \frac{1}{h^2(\bar{z}-z')^2+1}$$

It can be easily shown that

$$\lim_{z'\to\bar{z}}\left[W_1(\bar{z}-z')\right] = 1$$

By adding and subtracting $\dfrac{1}{(z - z')}$ equation (2.8-17) can be obtained in a form similar to equation (2.4-5).

$$\oint_0^1\frac{g_1(z')}{(\bar{z}-z')}dz' = -2\pi\alpha_r - \int_0^1 g_1(z')\left[\frac{W_1(\bar{z}-z') - 1}{(\bar{z}-z')} + \pi h\right]dz' \qquad (2.8\text{-}19)$$

where

$$\lim_{z'\to\bar{z}}\left[\frac{W_1(\bar{z}-z') - 1}{(\bar{z}-z')}\right] = 0$$

and this equation can be solved by the same procedure as given in Sections II.4 and II.6. Since the solution follows exactly that of the previous sections with only changes in the Kernel $K(\bar{z}, z')$ and in $f(\bar{z})$, a complete derivation will not be given here. Following the procedure of Section II.4 yields for equation (2.8-20) the following Fredholm equation of the second kind, where $\bar{z} = (\frac{1}{2})(1 + \cos\theta)$.

$$g_1^*(\theta) = (\sin\tfrac{1}{2}\theta)g_1(\theta) = 2\alpha\cos\tfrac{1}{2}\theta - \frac{2}{\pi}\int_0^{\pi}\left(\cos\tfrac{1}{2}\theta'\left[-b_0(\theta')\cos\tfrac{1}{2}\theta\right.\right.$$

$$\left.\left.+(\sin\tfrac{1}{2}\theta)\sum_{m=1}^{\infty}b_m(\theta')\sin m\theta\right]\right)g_1^*(\theta')d\theta' \tag{2.8-20}$$

where

$$b_0(\theta') = \frac{\pi}{2}h + \frac{1}{\pi}\int_0^{\pi}\left[\frac{W_1(\cos\theta''-\cos\theta')-1}{\cos\theta''-\cos\theta'}\right]d\theta'' = \frac{\pi h}{2} + \frac{1}{2}b_0^{(1)}(\theta') \tag{2.8-21}$$

$$b_m(\theta') = \frac{2}{\pi}\int_0^{\pi}\left[\frac{W_1(\cos\theta''-\cos\theta')-1}{(\cos\theta''-\cos\theta')}\right]\cos m\theta''d\theta'' \tag{2.8-22}$$

From Section II.6 the solution of (2.8-20) is obtained in the following form.

$$g_1^*(\theta) = 2\alpha_r\cos\tfrac{1}{2}\theta - \frac{2}{\pi}\sin\tfrac{1}{2}\theta\left[A_0\cot\tfrac{1}{2}\theta + A_1\sin\theta + A_2\sin 2\theta\right.$$

$$\left.+\cdots+A_m\sin m\theta\right] \tag{2.8-23}$$

The A_m's are obtained from a solution of the following set of algebraic equations

$$A_o(1-C_{oo}) - A_1C_{o1} - A_2C_{o2} - \cdots - A_mC_{om} = d_o$$

$$-A_oC_{1o} + A_1(1 - C_{11}) - A_2C_{12} \cdots - A_mC_{1m} = d_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-A_oC_{mo} - A_1C_{m1} - A_2C_{m2} - \cdots + A_m(1-C_{mm}) = d_m \tag{2.8-24}$$

The coefficients C_{mm} are

$$C_{mm} = C_{ij} = -\frac{1}{\pi}\int_0^{\pi} b_i(\theta')\,D_j(\theta')\sin\theta'd\theta', \quad (i,j = 0, 1, 2 \cdots m)$$

or more specifically

$$\begin{aligned}
C_{oo} &= +\frac{\pi}{2}h + \frac{1}{2\pi}\int_0^{\pi} b_o^{(1)}(\theta')(1+\cos\theta')d\theta' \\
C_{o1} &= -\frac{\pi}{4}h - \frac{1}{2\pi}\int_0^{\pi} b_o^{(1)}(\theta')\sin^2\theta'd\theta' \\
C_{oj} &= -\frac{1}{2\pi}\int_0^{\pi} b_o^{(1)}(\theta')\sin j\theta'\sin\theta'd\theta', \qquad (j = 2, 3, \cdots m) \\
C_{io} &= \frac{1}{\pi}\int_0^{\pi} b_i(\theta')(1+\cos\theta')d\theta', \qquad (i = 1,2,3 \cdots m) \\
C_{ij} &= -\frac{1}{\pi}\int_0^{\pi} b_i(\theta')\sin j\theta'\sin\theta'd\theta', \qquad (i,j = 1, 2, 3, \cdots m)
\end{aligned} \tag{2.8-25}$$

In the preceding equations for C_{ij} the coefficients $b_o^{(1)}(\theta)$ and $b_i(\theta)$ are given by equations (2.8-21) and (2.8-22) respectively. The coefficients d_m occurring in the set of equations (2.8-24) are dependent on the angle of incidence of the annular airfoil. They can be shown to be

$$d_m = d_i = \int_0^{\pi} \cos\tfrac{1}{2}\theta'\, b_i(\theta')\,(2\pi\alpha_r \cos\tfrac{1}{2}\theta')\,d\theta'$$

$$= 2\pi\alpha_r \int_0^{\pi} b_i(\theta')\cos^2\tfrac{1}{2}\theta'\,d\theta$$

$$= 2\pi\alpha_r f_i \qquad (i = 0, 1, 2, \cdots m) \tag{2.8-26}$$

where

$$f_0 = \frac{\pi^2}{4}h + \frac{i}{2}\int_0^{\pi} b_0^{(i)}(\theta')\cos^2\tfrac{1}{2}\theta'\,d\theta'$$

$$f_i = \int_0^{\pi} b_i(\theta')\cos^2\tfrac{1}{2}\theta'\,d\theta' \quad , \qquad (i = 1, 2, \cdots m) \tag{2.8-27}$$

As can be seen from the form of the coefficients, the set of equations for the A_m's are completely independent of the axial coordinate θ. Once the A_m's are determined from equation (2.8-24) they are substituted back into equation (2.8-23) for determining $g_1{*}(\theta)$ or as easily $g_1(\theta)$. The circulation distribution is readily calculated from equation (2.8-4), i.e.

$$\gamma(\varphi, \bar{z}) = g_0(\bar{z}) + g_1(\bar{z})\cos\varphi$$

$$= \gamma(\bar{z}) + g_1(\bar{z})\cos\varphi \tag{2.8-28}$$

In this section the linearized flow field about an annular airfoil has been derived and has been shown to be a linear combination of the airfoil at zero incidence and a term involving the angle of attack but not the section shape. It should be noted however that both terms are dependent on the chord-diameter ratio of the duct. Since the circulation term $g_1(\bar{z})$ is independent of the section shape, it can be tabulated for different angles of incidence and chord-diameter ratios.

The pressure and velocity distributions of the annular airfoil at an angle of incidence follow from Section II.7 and as can be seen from equation (2.8-28), the effect of the angle of attack is to add a term to the coefficient for the airfoil at zero incidence. For instance the linearized pressure distribution follows from equation (2.7-1), (A-15), (B-10) and (2.8-28) as.

$$\frac{p(\bar{z},R_d,\varphi) - p_0}{\frac{1}{2}\rho V_0^2} = 2\left[\frac{w_a}{V_0}(\bar{z},R_d)\right]_\gamma + 2\left[\frac{w_a}{V_0}(\bar{z},R_d)\right]_q$$

$$\pm g_1(z)\cos\varphi - \frac{h}{\pi}\int_0^1 \frac{g_1(z')k^3}{8}\left[\int_0^{2\pi}\frac{[\cos(\varphi-\varphi')-1]\cos\varphi'}{[1-k^2\cos^2\frac{1}{2}(\varphi-\varphi')]^{3/2}}d\varphi'\right]dz'$$

$$= 2\left[\frac{w_a}{V_0}(\bar{z},R_d)\right]_\gamma + 2\left[\frac{w_a}{V_0}(\bar{z},R_d)\right]_q + (\cos\varphi)\left[P(\bar{z}) \pm g_1(\bar{z})\right] \tag{2.8-29}$$

where

$\frac{w_a}{V_0}(\bar{z},R_d)_\gamma$ = induced velocity from vortex distribution of the annular airfoil at zero incidence, equation (A-22).

$\frac{w_a}{V_s}(\bar{z},R_d)_q$ = induced velocity from source distribution, equation (B-10).

$$P(z) = -\frac{h}{\pi}\int_0^1 \frac{g_1(z')}{k}\left\{(1-k^2)\left[E(k)-3K(k)\right]+3E(k)-K(k)\right\}dz'$$

The integrand of the integral has a logarithmic singularity at $k = 1$ and a square root singularity at $z' = 1$. To remove

these first the variables $z' = (\frac{1}{2})(1 + \cos \theta')$ and $\bar{z} = (\frac{1}{2})(1 + \cos \bar{\theta})$ are introduced, then the variable $(\cos \bar{\theta} - \cos \theta') = \cos^3\theta$,

$$P(\bar{z}) = \frac{3h}{\pi} \int_{\arccos(-\sqrt[3]{2(1-\bar{z})})}^{\arccos(\sqrt[3]{2\bar{z}})} \frac{g_1^*(\theta')}{k} \Big[(4-k^2)E(k) - (4-3k^2)K(k)\Big] \cos^2\theta' \cos\tfrac{1}{2}\theta' \, d\theta' \tag{2.8-30}$$

where

$$k^2 = \frac{4}{h^2\cos^6\theta' + 4}$$

The lift of the annular airfoil at an angle of incidence is given by equation (2.7-6) and equation (2.8-28).

$$C_L = \frac{L}{\frac{\rho}{2} V_0^2 a R_d} = -2 \int_0^{2\pi} \left((\cos\varphi) \int_0^1 \Big[g_0(z') + g_1(z')\cos\varphi \Big] dz' \right) d\varphi$$

$$= -2 \int_0^1 g_1(z') \Big[\int_0^{2\pi} \cos^2\varphi \, d\varphi \Big] dz'$$

$$= -2\pi \int_0^1 g_1(z') dz' = -2\pi \int_0^{\pi} g_1^*(\theta) \cos\tfrac{1}{2}\theta \, d\theta \tag{2.8-31}$$

The induced drag follows from equations (2.7-8), (2.8-28) and (C-12) as

$$C_{Di} = -\frac{T_d}{\frac{\rho}{2} V_0^2 a R_d} = -2 \int_0^1 \int_0^{2\pi} \Big[g_0(\bar{z}) + g_1(\bar{z})\cos\varphi \Big] \Big[\frac{W_r}{V_0}(\varphi, R_d, \bar{z}) \Big]_{\frac{\partial\gamma}{\partial\varphi}} d\varphi \, d\bar{z}$$

$$= -h \int_0^1 g_1(\bar{z}) \left(\int_0^1 g_1(z') \Big[\pi + \frac{4h(\bar{z}-z')}{k} \big[K(k) - E(k) \big] \Big] dz' \right) d\bar{z}$$

$$C_{a_i} = -h\int_0^{\pi} g_1^*(\theta)\cos\tfrac{1}{2}\theta\left(\int_0^{\pi} g_1^*(\theta')\cos\tfrac{1}{2}\theta'\left[\pi\right.\right.$$

$$\left.\left. + \frac{2h(\cos\theta - \cos\theta')}{k}\left[K(k) - E(k)\right]\right]d\theta'\right)d\theta \qquad (2.8\text{-}32)$$

where

$$k^2 = \frac{4}{h^2(\cos\theta - \cos\theta')^2 + 4}$$

The moment on the annular airfoil about the leading edge is

$$C_M = \frac{M}{\frac{\rho}{2}a^2V_0^2R_d} = \int_0^{2\pi}(\cos\varphi)\left[\int_0^1 \gamma(\varphi,\bar{z})(\bar{z}-1)\,d\bar{z}\right]d\varphi \qquad (2.8\text{-}33)$$

Introducing the Fourier series for γ and the variable $\bar{z} = (\frac{1}{2})(1 + \cos\theta)$, the moment can be shown to depend only on angle of attack and not the section shape.

$$C_M = -\frac{\pi}{2}\int_0^{\pi} g_1^*(\theta)\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,d\theta \qquad (2.8\text{-}34)$$

Some of these coefficients have been tabulated by Weissinger[19].

III. Effect of the Hub

Assuming that the ordinate of the surface of the hub is denoted by $x_h(z)$ and a polar coordinate system attached to a propeller blade is used, the linearized kinematic boundary condition which must be satisfied on the hub follows directly from section II.2 as

$$\frac{w_r}{V_s}(x_h, \varphi, z) = -(1 - w_{x_h})\, x_h'(z) \tag{3-1}$$

As before the velocities have been nondimensionalized by the ship speed and the radial coordinate by the propeller diameter. The function $x_h'(z)$ is the slope of the hub surface

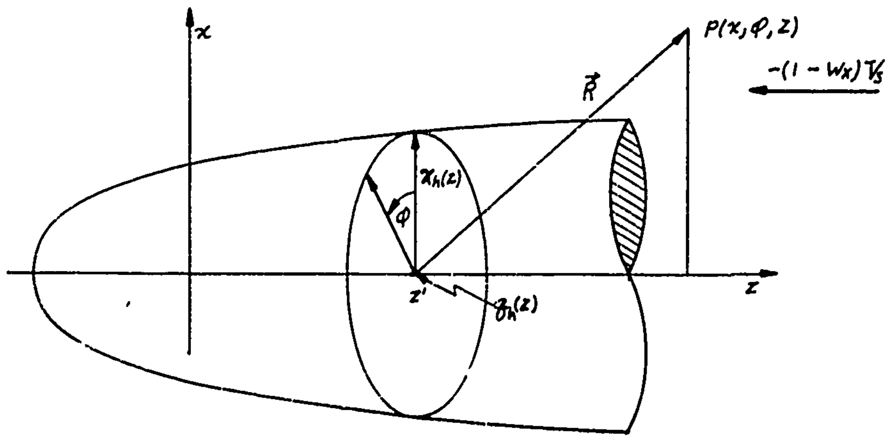

Figure 4. Notation for the hub

Since the hub is a symmetrical body and is assumed to have no angle of attack with respect to the free-stream velocity, the shape of the hub can be represented by a distribution of

sources, doublets or vortices[30] over the surface of the hub. If the radial velocity can be shown to be independent of angle and the hub is not too blunt, then the flow can be represented by a distribution of sources and sinks along the z-axis only, as indicated in Figure 4. The velocity $w_r(x_h, \varphi, z)$ in equation (3-1) represents the whole flow field, i.e. it is composed of the radial velocity induced by the annular airfoil with its trailing vortex system, the propeller and the hub itself. In terms of the nondimensionalized velocities of the individual singularities equation (3-1) can be rewritten as follows

$$\frac{w_r}{V_s}(x_h, \varphi, z)_{hub} = -(1 - w_{x_h})x_h'(z) - \frac{w_r}{V_s}(x_h, \varphi, z)_\gamma - \frac{w_r}{V_s}(x_h, \varphi, z)_g$$

$$- \frac{w_r}{V_s}(x_h, \varphi, z)_{\frac{\partial\gamma}{\partial\varphi}} - \frac{w_r}{V_s}(x_h, \varphi, z)_p$$

$$= -(1 - w_{x_h})x_h'(z) - \frac{w_r}{V_s}(x_h, \varphi, z)_d - \frac{w_r}{V_s}(x_h, \varphi, z)_p \qquad (3\text{-}2)$$

Since normally it would be expected that the radial velocity induced by the propeller and the annular airfoil at the hub is small, it will be assumed that the hub source distribution satisfies only the average velocity at the hub. If the hub radius is zero, this must be satisfied exactly. Consequently the radial velocities induced at the hub will be expressed as follows:

$$\frac{w_r}{V_s}(x_h, z)_d = \frac{1}{2\pi}\int_0^{2\pi} \frac{w_r}{V_s}(x_h, \varphi, z)_d \, dz \qquad (3\text{-}3)$$

and

$$\frac{w_r}{V_s}(x_h,z)_p = \frac{1}{2\pi}\int_0^{2\pi} \frac{w_r}{V_s}(x_h,\varphi,z)_p \, dz$$

The use of these induced velocities means essentially that only the average radial velocity at the hub is considered and that now the hub can mathematically be represented by a source distribution along the z-axis. Using equation (3-3) the boundary condition equation (3-2), now becomes

$$\frac{w_r}{V_s}(x_h,z)_{hub} = -(1 - w_{x_h})\lambda_h'(z) - \frac{w_r}{V_s}(x_h,z)_d - \frac{w_r}{V_s}(x_h,z)_p \qquad (3\text{-}4)$$

For the derivation of the induced velocities from a distribution of sources and sinks along a line, consider first a single three-dimensional source of strength $q_h(\xi')$ at the point $\xi = \xi'$, r=0. In polar coordinates the stream function of such a point source is[13] (see Figure 4)

$$\psi(r,\xi) = -\frac{q_h(\xi')}{4\pi}\left(1 + \frac{\xi - \xi'}{[(\xi - \xi')^2 + r^2]^{1/2}}\right) \qquad (3\text{-}5)$$

The stream function of an axisymmetric body is obtained by integrating a distribution of sources of strength $q_h(z')$ per unit length along the ξ-axis from the after end ($\xi = b_1$) to the nose ($\xi = b_2$).

$$\psi(r,\xi)_b = -\frac{1}{4\pi}\int_{b_1}^{b_2} q_h(\xi')\left(1 + \frac{\xi - \xi'}{[(\xi - \xi')^2 + r^2]^{1/2}}\right) d\xi' \qquad (3\text{-}6)$$

The induced velocities follow from equation (2.2-10) by

differentiating this equation.

$$w_a(r,\xi)_h = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{1}{4\pi}\int_{b_1}^{b_2} g_h(\xi')\left(\frac{\xi-\xi'}{[(\xi-\xi')^2+r^2]^{3/2}}\right)d\xi' \tag{3-7}$$

$$w_r(r,\xi)_h = -\frac{1}{r}\frac{\partial\psi}{\partial\xi} = \frac{r}{4\pi}\int_{b_1}^{b_2}\frac{g_h(\xi')\,d\xi'}{[(\xi-\xi')^2+r^2]^{3/2}} \tag{3-8}$$

If the induced velocity is nondimensionalized by the ship speed (V_s) the source stren th by the ship speed times the propeller tip circumference ($2\pi R_p V_s$), the axial coordinate ξ , by the duct chord (a) and the radial coordinates by the propeller radius (R_p), the induced velocities can be written in nondimensionalized form as in section II. This source strength has dimensions of length^2 per unit time compared to the elementary strength of the ring vortex which has the dimensions of length per unit time.

$$\frac{w_a}{V_s}(x,z)_h = \frac{h}{x_d}\int_{b_1}^{b_2} g_h(z')\frac{2h(z-z')\,dz'}{[4h^2(z-z')^2+(\frac{x}{x_d})^2]^{3/2}} \tag{3-9}$$

$$\frac{w_r}{V_s}(x,z)_h = \frac{h}{x_d}\left(\frac{x}{x_d}\right)\int_{b_1}^{b_2}\frac{g_h(z')\,dz'}{[4h^2(z-z')^2+(\frac{x}{x_d})^2]^{3/2}} \tag{3-10}$$

where

$$h = \frac{\text{duct chord}}{\text{duct diameter}} = \frac{a}{2x_d R_p}$$

at the propeller z=o and the equation (3-9) becomes

$$\frac{w_a}{V_s}(x,0)_h = -\frac{h}{x_d}\int_{b_1}^{b_2}\frac{2hz'g_h(z')\,dz'}{[4h^2z'^2+(\frac{x}{x_d})^2]^{3/2}} \tag{3-11}$$

Sometimes it may be more convenient to nondimensionalize the axial coordinate by the propeller diameter, i.e. let $z_p = \frac{\xi}{R_p}$, then the axial induced velocity can be written as follows

$$\frac{w_a}{V_s}(x,0)_h = -\frac{1}{2}\int_{b_1}^{b_2} \frac{z_p' \, q_h(z_p')}{\left[z_p'^2 + \left(\frac{x}{x_d}\right)^2\right]^{3/2}} \, dz_p' \qquad (3\text{-}12)$$

The problem now is to find the source distribution $q_h(z')$ which represents the shape of the hub. For this the boundary condition given by (3-4) must be used. From equation (3-10) the radial velocity at the hub is

$$\frac{w_r}{V_s}(x_h,z)_h = \frac{h}{x_d}\left(\frac{x_h}{x_d}\right)\int_{b_1}^{b_2} \frac{q_h(z')\, dz'}{\left[4h^2(z-z')^2 + \left(\frac{x_h}{x_d}\right)^2\right]^{3/2}}$$

and then the boundary condition becomes

$$\frac{h}{x_d}\left(\frac{x_h}{x_d}\right)\int_{b_1}^{b_2} \frac{q_h(z')\, dz'}{\left[4h^2(z-z')^2 + \left(\frac{x_h}{x_d}\right)^2\right]^{3/2}} = -(1 - w_{x_h})\, x_h'(z) - \frac{w_r}{V_s}(x_h,z)_p - \frac{w_r}{V_s}(x_d,z)_d \qquad (3\text{-}13)$$

This is a Fredholm integral equation of the first kind for the unknown source distribution $q_h(z)$. Direct inversion of this type of integral equation usually is not possible and a solution normally involves an infinite series of eigenvalues of the kernal and eigenfunctions. An iteration procedure for the solution of this type of integral equation has been developed by Landweber[31]. In general, since the diameter of the hub is normally small compared to its length (the hub is often assumed

to be infinite in length), and the velocity distribution some distance from the hub and not on the hub itself is desired, further simplification can reasonably be made. The previous statement about the effect of the hub essentially describes the assumptions involved in slender body theory[30] and it is this theory which will be applied now.

Laitone[32] derived the slender body theory by expanding the source distribution in a Taylor series. The first term of this series gives the result of slender body theory. Applying the result from Reference [32] directly to equation (3-13), the following equation for the source strength $q_h(z)$ results.

$$\frac{q_h(z)}{x_h(z)} = -(1 - w_{x_h})\, x_h'(z) - \frac{w_r}{V_s}(x_h, z)_p - \frac{w_r}{V_s}(x_h, z)_d \tag{3-14}$$

or

$$q_h(z) = -(1 - w_{x_h})\frac{A'(z)}{2\pi} - x_h(z)\left[\frac{w_r}{V_s}(x_h, z)_p + \frac{w_r}{V_s}(x_h, z)_d\right]$$

where $A(z) = \pi x_h^2(z)$, i.e. the hub cross-sectional area.

Equation (3-14) shows that as a first approximation the source strength representing the hub at a point is a function of the change in the cross-sectional area of the hub at that point and the radial velocity induced at the hub surface by the annular airfoil and propeller blades. The velocity induced by the hub at any point in the surrounding flow field is obtained by substituting equation (3-12) into (3-9) and (3-10). If the hub is of constant diameter, then the slope is zero

$(\alpha_h'(z)=0)$ and the source strength is a function of only the velocities induced at the hub by the duct and propeller. If the hub is assumed to be infinitely long, then $b_1=-\infty$ and $b_2 = \infty$ in equations (3-9) and (3-10). From a practical point of view this presents no problem since the point for which the induced velocity is desired (x,z) will be at or close to the propeller and therefore at large values of z the integrands of equations (3-9) and (3-10) become small very rapidly, as $\frac{1}{z^2}$ for the axial induced velocity and $\frac{1}{z^3}$ for the radial induced velocity, and the integral can be shown to converge uniformly. Furthermore the slope of the hub must be either zero or undulate some distance from the propeller. Consequently the hub shape some distance from the propeller has no affect on the flow through the propeller. It would be expected that in the normal case the velocity induced by the hub at the duct would be negligible.

IV. Circulation Theory of the Propeller in the Duct

IV.1 Introduction and Assumptions

The circulation theory as applied to propellers is analogous to lifting-line theory[27] of wings of finite length. The main difference, which considerably complicates the flow field, is that the trailing vortices which were assumed to lie lay in the plane of the wing now lie along helices. In the rotating coordinate system these helicoidal vortex sheets are stream surfaces s are the vortex sheets behind wings.

Lerbs has developed the theory of the moderately loaded propeller[33] and his general approach will be used here.* In this theory Lerbs considers the influence of the induced velocities on the shape of the helical vortex sheet at the lifting line but neglects the effects of centrifugal force and of the contraction of the slip-stream. In addition, the change in shape of the vortex lines are neglected in the axial direction, i.e. they are of constant pitch. These same assumptions will be made here and further it will be assumed that the influence of the duct on the change in the shape of the helical vortex sheet in the axial direction can be neglected. It should be mentioned here that the vortex sheets are not necessarily true helicoidal surfaces since the pitch may vary along the radius but each vortex line is assumed to be of constant pitch.

*Many of the derivations given in this chapter follow closely unpublished class notes of Professor J.V.Wehausen on "Hydrodynamics of Ships," University of California, Berkeley.

In addition to the effect of the duct on the propeller, the radial velocities induced on the cylinder representing the duct by the propeller must also be considered. For this reason the induced radial velocity of the propeller must be derived in more general terms than was done by Lerbs.

In the following development the free-stream velocity will be allowed to have a radial variation but must be axisymmetric and the propeller may take any axial position in relation to the duct. Th_ other major assumptions are stated more explicitly in Section I.

The boundary conditions imposed on the bound circulation is that it be zero at the hub,[33] $\Gamma(R_h)=0$ and if $R_d > R_p$ then the circulation at the blade tip is zero, $\Gamma(R_p)=0$. If the diameter of the duct and propeller are equal ($R_d=R_p$), then the circulation at the tip need not be zero. This last statement comes from the fact that no equalization of pressure takes place around the blade tip if there is no clearance and the duct is sufficiently long. From a practical point of view, because of the boundary layer, no equalization of pressure will take place if the tip clearance is small. The determination of how small is sufficiently small requires an analysis from boundary layer theory. This is not treated here.

IV.2 Induced Velocities from the Vortex Lines of one Propeller Blade

The flow field of each propeller blade is considered to be made up of a system of horse-shoe vortices lying along a helix. The elementary system used will consist of three parts; a single helical vortex line, the bound vortex lying along a radius, and another free vortex line along the negative ξ -axis. This system is shown in the following figure.

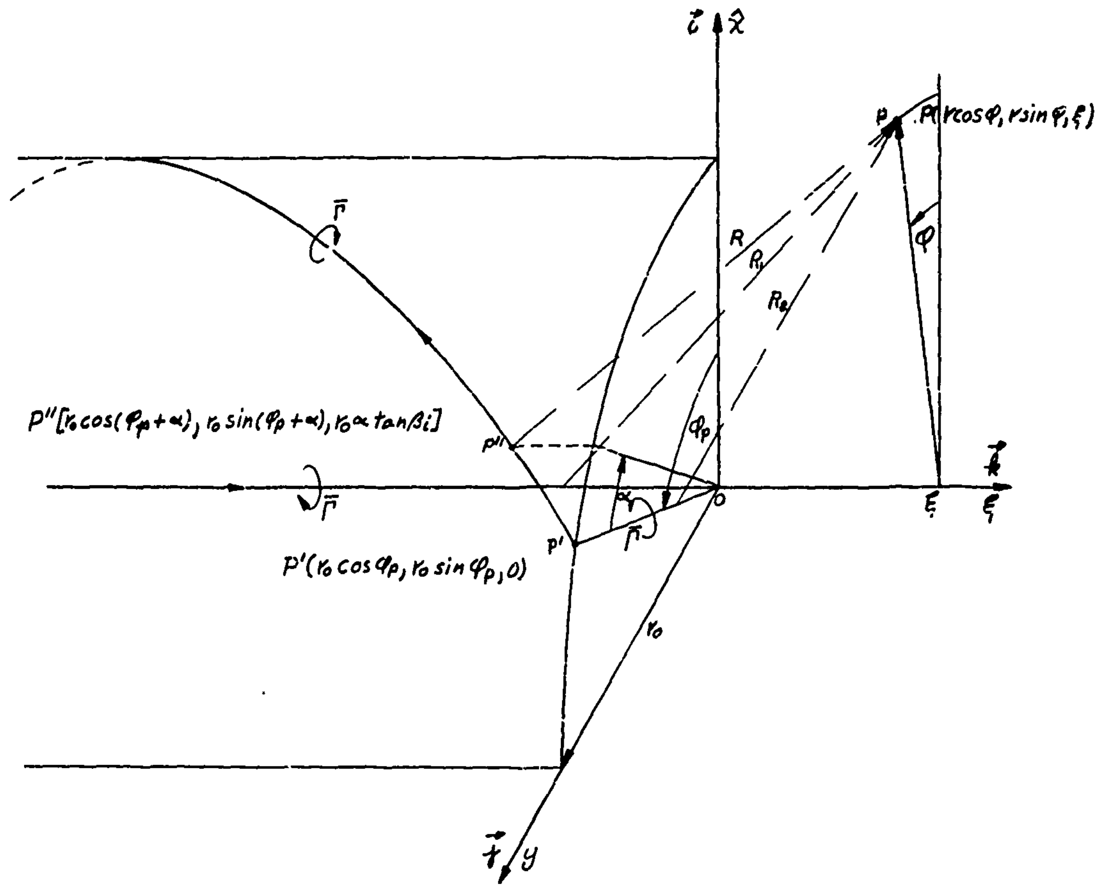

Figure 6. Propeller vortex system

The point P, is the point at which the induced velocity is desired. It is assumed that one blade is located on the $\hat{x}$-axis and another at the angle φ_p. Letting (b) be the number of blades the angle φ_p is given by

$$\varphi_p = (p-1)\frac{2\pi}{b} \quad , \quad p = 1, 2, \ldots, b \tag{4.2-1}$$

The location of point P is taken as arbitrary. Only the radial velocity is needed to satisfy the boundary condition on the hub and duct so only the radial velocity will be calculated at the arbitrary point P ($r \cos \varphi$, $r \sin\varphi$, ξ). The axial and tangential velocities induced by this system are needed only at each blade. Since the blades are assumed to be identical, it suffices to consider only the blade along the $\hat{x}$-axis, i.e. at the point P(x,0,0) = P(r,0,0).

In Figure 6 the other singularities representing the hub and duct could be considered but since the strength of these singularities are dependent on the induced velocities from the propeller vortex system this procedure is not practical and a method of iteration must be used. The other singularities affect the hydrodynamic pitch angle β_i shown in this figure.

The velocity induced by a single helical vortex line follows from the Biot-Savart law, and is derived in Appendix D. Designating this contribution to the induced velocity by a superscript (1), the radial velocity induced at an arbitrary point P by a helical vortex line leaving from the point P'($r_o \cos \varphi_p$, $r_o \sin \varphi_p$,0) follows from equation (D.8) as

$$w_r^{(1)} = \frac{\bar{\Gamma} r_0}{4\pi} \int_{-\infty}^{0} \left(\frac{\xi \cos(\varphi - \varphi_p - \alpha) - r_0 [\sin(\varphi - \varphi_p - \alpha) + \alpha \cos(\varphi - \varphi_p - \alpha) \tan\beta_i}{[r^2 + r_0^2 + (z - r_0 \alpha \tan\beta_i)^2 - 2 r r_0 \cos(\varphi - \varphi_p - \alpha)]^{3/2}} \right) d\alpha \tag{4.2-2}$$

The axial and tangential velocities induced on the propeller blade at P(r,0,0) by the same vortex line is obtained from equations (D-7) and (D-9).

$$w_a^{(1)} = \frac{\bar{\Gamma} r_0}{4\pi} \int_{-\infty}^{0} \left(\frac{r_0 - r \cos(\varphi_p + \alpha)}{[r^2 + r_0^2 - 2 r r_0 \cos(\varphi_p + \alpha) + r_0^2 \alpha^2 \tan^2\beta_i]^{3/2}} \right) d\alpha \tag{4.2-3}$$

$$w_t^{(1)} = \frac{\bar{\Gamma} r_0 \tan\beta_i}{4\pi} \int_{-\infty}^{0} \left(\frac{r - r_0 \cos(\varphi_p + \alpha) - r_0 \alpha \sin(\varphi_p + \alpha)}{[r^2 + r_0^2 - 2 r r_0 \cos(\varphi_p + \alpha) + r_0^2 \alpha^2 \tan^2\beta_i]^{3/2}} \right) d\alpha \tag{4.2-4}$$

The velocity induced by the vortex line along the z-axis again follows from the law of Biot-Savart. Using the designation superscript (2) the induced velocity is

$$\vec{V}_i^{(2)} = \frac{\bar{\Gamma}}{4\pi} \int_{-\infty}^{0} \frac{\vec{R}_1 \times d\vec{s}}{R_1^3} = \frac{\bar{\Gamma}}{4\pi} \int_{-\infty}^{0} \frac{[(r\cos\varphi)\vec{i} + (r\sin\varphi)\vec{j} - \xi \vec{k}] \times [0\vec{i} + 0\vec{j} + \vec{k}]}{[r^2\cos^2\varphi + r^2 \sin^2\varphi + \xi^2]^{3/2}} d\xi$$

$$= \frac{\bar{\Gamma}}{4\pi} \int_{-\infty}^{0} \frac{[(r\sin\varphi)\vec{i} - (r\cos\varphi)\vec{j}]}{[r^2 + \xi^2]^{3/2}} d\xi = -\frac{\bar{\Gamma}}{4\pi} \left[\frac{(r\sin\varphi)\vec{i} - (r\cos\varphi)\vec{j}}{r^2} \right] \tag{4.2-5}$$

From this equation it can be seen that the induced axial velocity from the vortex along the z-axis is zero i.e. ($w_a^{(2)}=0$) and also, as would be expected, the induced radial velocity is zero. This follows from equations (4.2-5) and (A-11).

$$w_r^{(2)} = -\frac{\bar{\Gamma}}{4\pi r^2} (-r \sin\varphi \cos\varphi - r\cos\varphi \sin\varphi) = 0$$

The tangential velocity follows from the same set of

equations and is

$$w_t^{(2)} = -\frac{\Gamma}{4\pi r^2}\left(-r\sin^2\varphi - r\cos^2\varphi\right) = \frac{\Gamma}{4\pi r}$$

(4.2-6)

The induced velocity from the radial vortex line (lifting line) is also obtained from the Biot-Sevart law and is

$$\vec{V}_i^{(3)} = -\frac{\Gamma}{4\pi}\int_0^{r_0}\left(\frac{[(r\cos\varphi - r'\cos\varphi_p)\vec{i} + (r\cos\varphi - r'\sin\varphi_p)\vec{j} + \xi\vec{k}]\times[\cos\varphi_p\vec{i} + \sin\varphi_p\vec{j} + 0\vec{k}]}{[\xi^2 + r^2 + r'^2 - 2rr'\cos(\varphi-\varphi_p)]^{3/2}}\right)dr'$$

$$= -\frac{\Gamma}{4\pi}\int_0^{r_0}\left(\frac{(-\xi\sin\varphi_p)\vec{i} + (\xi\cos\varphi_p)\vec{j} - [r\sin(\varphi-\varphi_p)]\vec{k}}{[\xi^2 + r^2 + r'^2 - 2rr'\cos(\varphi-\varphi_p)]^{3/2}}\right)dr' \qquad (4.2\text{-}7)$$

$$= \frac{-\Gamma}{4\pi[\xi^2 + r^2\sin^2(\varphi-\varphi_p)]}\left[\frac{r_0 - r\cos(\varphi-\varphi_p)}{\sqrt{\xi^2 + r^2 + r_0^2 - 2rr_0\cos(\varphi-\varphi_p)}} + \frac{r\cos(\varphi-\varphi_p)}{\sqrt{(\xi^2 + r^2)}}\right]$$

$$\cdot\left[(-\xi\sin\varphi_p)\vec{i} + (\xi\cos\varphi_p)\vec{j} - [r\sin(\varphi-\varphi_p)]\vec{k}\right] \qquad (4.2\text{-}8)$$

The axial induced velocity of the lifting line is given by the component in the $\vec{k}$ direction.

$$W_a^{(3)} = \frac{\Gamma r\sin(\varphi-\varphi_p)}{4\pi[\xi^2 + r^2\sin(\varphi-\varphi_p)]}\left[\frac{r_0 - r\cos(\varphi-\varphi_p)}{\sqrt{\xi^2 + r^2 + r_0^2 - 2rr_0\cos(\varphi-\varphi_p)}} + \frac{r\cos(\varphi-\varphi_p)}{\sqrt{(\xi^2 + r^2)}}\right]$$

$$,(\varphi_p \neq \varphi \pm n\pi,\ n = 0,1,2\ldots) \qquad (4.2\text{-}9)$$

For $\varphi_p = \varphi \pm n\pi$, (n=0,12...) it can be shown that $w_a^{(3)}=0$. At the reference blade φ =0 and z=0 and the above equation reduces to

$$W_a^{(3)} = \frac{-\Gamma}{4\pi r\sin\varphi_p}\left[\frac{r_0 - r\cos\varphi_p}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\varphi_p}} + \cos\varphi_p\right] \quad ,(\varphi_p \neq n\pi,\ n=0,1,2\ldots)$$

(4.2-10)

and if $\varphi_p = \pm n\pi$, $(n=0,1,2\ldots)$ $w_a^{(3)}=0$

The radial induced velocity for an arbitrary point $P(r, \varphi, z)$ is obtained from equation (4.2-8) as

$$w_r^{(3)} = \frac{-\bar{\Gamma}[\xi \sin(\varphi-\varphi_p)]}{4\pi[\xi^2 + r^2\sin^2(\varphi-\varphi_p)]}\left[\frac{r_0 - r\cos(\varphi-\varphi_p)}{\sqrt{\xi^2+r^2+r_0^2-2rr_0\cos(\varphi-\varphi_p)}} + \frac{r\cos(\varphi-\varphi_p)}{\sqrt{(\xi^2+r^2)}}\right] \quad (4.2\text{-}11)$$

For $\xi = 0$ or $\varphi = \varphi_p \pm n\pi$, $(n=0,1,2\ldots)$, $w_r^{(3)}=0$

The tangential velocity at the propeller blade is zero ($w_t^{(3)}=0$). This is obvious since at the blade $\xi=0$ and by equation (4.2-8) both the $\vec{i}$ and $\vec{j}$ components are then zero. No singularities occur in equations (4.2-10) and (4.2-11). This can be seen by examination of equation (4.2-7).

The induced velocities at a point P from the three vortex lines is obtained by summing and at the blade, P(r,0,0), are

$$(w_a) = w_a^{(1)} + w_a^{(3)},\ w_a^{(2)}=0 \quad (4.2\text{-}12)$$

where $w_a^{(1)}$ is given by equation (4.2-3) and $w_a^{(3)}$ is given by equation (4.2-9)

$$(w_t) = w_t^{(1)} + w_t^{(2)},\ w_t^{(3)}=0 \quad (4.2\text{-}13)$$

where $w_t^{(1)}$ is given by equation (4.2-4) and $w_t^{(2)}$ by equation (4.2-6).

The radial velocity induced at an arbitrary point $P(r\sin\varphi, r\cos\varphi, \xi)$ is given by

$$(w_r)_p = w_r^{(1)} + w_r^{(3)} \quad , \quad w_r^{(2)} = 0 \tag{4.2-14}$$

where $w_r^{(1)}$ is given by equation (4.2-2) and $w_r^{(3)}$ is given by equation (4.2-11).

IV.3 Induced Velocities from the Vortex Lines from All the Blades.

As discussed previously the blades are assumed to be evenly spaced with one vertically upward along the positive $\hat{x}$-axis and the helical vortex lines shed at a given radius are assumed to leave all the blades at the same pitch. The velocity induced at one of the blades by the horseshoe vortices from all the blades is desired. It is sufficient for this purpose to consider the blade along the vertical $\hat{x}$-axis as the reference blade and calculate the velocities induced at a point P(r, 0, 0) on this blade. Also desired besides the velocity induced at a blade is the radial velocity at an arbitrary point $P(r\cos\varphi, r\sin\varphi, \xi)$.

First consider the velocity components induced by the helical vortex lines shed from the blades at the radius r_o. The total contribution to the axial and tangential components at any one blade is obtained by summing equations (4.2-3) and (4.2-4) over the number of blades.

$$(w_a^{(1)})_{b_o} = \frac{\bar{\Gamma} r_o}{4\pi} \sum_{p=1}^{b} \int_{-\infty}^{0} \left[\frac{r_o - r\cos(\varphi_p + \alpha)}{[r^2 + r_o^2 - 2rr_o\cos(\varphi_p + \alpha) + r_o^2\alpha^2\tan^2\beta_i]^{3/2}} \right] d\alpha \tag{4.3-1}$$

$$(w_t^{(1)})_{b_o} = \frac{\bar{\Gamma} r_o \tan\beta_i}{4\pi} \sum_{p=0}^{b} \int_{-\infty}^{0} \left[\frac{r - r_o[\cos(\varphi_p + \alpha) + \sin(\varphi_p + \alpha)]}{[r^2 + r_o^2 - 2rr_o\cos(\varphi_p + \alpha) + r_o^2\alpha^2\tan^2\beta_i]^{3/2}} \right] d\alpha \tag{4.3-2}$$

The radial velocity induced at an arbitrary point $P(r\cos\varphi, r\sin\varphi, \xi)$ by all the vortices shed from r_o is obtained from equation (4.2-2) as

$$(w_r^{(1)})_{b_0} = \frac{\bar{\Gamma} r_0}{4\pi} \sum_{p=1}^{b} \int_{-\infty}^{0} \left[\frac{\xi \cos(\varphi - \varphi_p - \alpha) - r_0[\sin(\varphi - \varphi_p - \alpha) + \alpha \cos(\varphi - \varphi_p - \alpha)] \tan\beta_i}{[r^2 + r_0^2 + (\xi - r_0 \alpha \tan\beta_i)^2 - 2 r r_0 \cos(\varphi - \varphi_p - \alpha)]^{3/2}} \right] d\alpha \tag{4.3-3}$$

The velocity induced by the hub vortex consists only of the tangential component. Summing this component results in equation (4.2-6) being multiplied by the number of blades, i.e.

$$(w_t^{(2)})_{b_0} = \frac{b\bar{\Gamma}}{4\pi r} \tag{4.3-4}$$

Finally the velocity components induced by the vortex along the lifting line will be discussed. At the reference blade itself only the axial component will be considered as on the blade both the tangential and radial velocities are zero. Referring to equation (4.2-10) it can be shown that there is no contribution from this component either, i.e. $(w_a^{(3)})_{b_0} = 0$. In the first place there is no contribution from the reference blade itself nor from a blade opposite it, i.e. for n = 0 and 1. The effect of the other blades must cancel in pairs since $\varphi_b = -\varphi_2, \varphi_{b-1} = -\varphi_3$, etc. Consequently, since $\sin \varphi_p = -\sin(-\varphi_p)$ and $\cos\varphi_p = \cos(-\varphi_p)$, it follows when equation (4.2-9) is summed over the number of blades that $(w_a^{(3)})_{b_0}=0$.

The radial velocity induced at an arbitrary point by the radial vortex lines equation (4.2-11) does not cancel. This velocity follows by summing equation (4.2-10) over the number of blades.

$$(w_r^{(3)})_{b_o} = \frac{-\bar{\Gamma}\xi}{4\pi}\sum_{p=1}^{b}\frac{\sin(\varphi-\varphi_p)}{[\xi^2 + r^2\sin^2(\varphi-\varphi_p)]}\left[\frac{r_o - r\cos(\varphi-\varphi_p)}{\sqrt{\xi^2 + r^2 + r_o^2 - 2rr_o\cos(\varphi-\varphi_p)}} + \frac{r\cos(\varphi-\varphi_p)}{\sqrt{\xi^2 + r^2}}\right]$$

(4.3-5)

For $\xi = 0$ or $\varphi = \varphi_p + n\pi$, $(n = 0, 1, 2, \ldots)$, $(w_r^{(3)})_{b_o} = 0$.

The integrands of the integrals for the axial and tangential velocities on the blade, equations (4.3-1) and (4.3-2), are singular. To facilitate numerical calculation of these integrals the "induction factors"[33] are introduced. These are nondimensional qualities of the induced velocity components and are a function of geometry only. They are defined by

$$(w_a^{(1)})_{b_o} = \frac{\bar{\Gamma}}{4\pi(r-r_o)}\, i_a \qquad \text{and} \qquad (w_a^{(t)})_{b_o} = \frac{\bar{\Gamma}}{4\pi(r-r_o)}\, i_t$$

(4.3-6)

From equations (4.2-3) and (4.2-4) it follows that the axial and tangential induction factors at the blades are

$$i_a\left(\frac{r_o}{r}\right) = \left(\frac{r_o}{r}\right)\left(1 - \frac{r_o}{r}\right)\sum_{p=1}^{b}\int_{-\infty}^{0}\left[\frac{\frac{r_o}{r} - \cos(\varphi_p+\alpha)}{\left[1 + \left(\frac{r_o}{r}\right)^2 - 2\left(\frac{r_o}{r}\right)\cos(\varphi_p+\alpha) + \left(\frac{r_o}{r}\right)^2\alpha^2\tan^2\beta_i\right]^{3/2}}\right]d\alpha$$

(4.3-7)

$$i_t\left(\frac{r_o}{r}\right) = \left(\frac{r_o}{r}\right)\left(1 - \frac{r_o}{r}\right)\tan\beta_i\sum_{p=1}^{b}\int_{-\infty}^{0}\left[\frac{1 - \left(\frac{r_o}{r}\right)\left[\cos(\varphi_p+\alpha) - \alpha\sin(\varphi_p+\alpha)\right]}{\left[1 + \left(\frac{r_o}{r}\right)^2 - 2\left(\frac{r_o}{r}\right)\cos(\varphi_p+\alpha) + \left(\frac{r_o}{r}\right)^2\alpha^2\tan^2\beta_i\right]^{3/2}}\right]d\alpha$$

(4.3-8)

In the limit as $r \rightarrow r_o$

$$\lim_{r \to r_o} i_a\left(\frac{r_o}{r}\right) = \cos\beta i$$

and

$$\lim_{r \to r_o} i_t\left(\frac{r_o}{r}\right) = \sin\beta i$$

The induction factors have been tabulated by Morgan[34] and are given in graphical form in Lerbs' paper.[33] Details of the method of calculating the induction factors are given by Lerbs[33] and Wrench.[35].

The total velocity indu ed by (b) vortex lines is obtained by summing the different velocity components. This summing does not represent the total velocity induced on a blade by the system of vortices but only from a single horse-shoe vortex on each blade. To obtain the total velocity induced at a point from all the horse-shoe vortices the summed equations must be integrated from the hub to the tip of the blade. The circulation strength Γ of a single trailing vortex is related to the total circulation along the blade. From wing theory[27] it follows that the strength of this elementary vortex is $\frac{d\Gamma}{dr_o}$ where $\Gamma(r_o)$ is the total strength of the bound vortex on the blade. With this notation it then follows from equation (4.3-6) that the axial velocity induced by the propeller at one of its blades is

$$w_a(r)_p = \frac{1}{4\pi}\int_{R_b}^{R_p} \frac{d\Gamma(r_o)}{dr_o}\,\frac{i_a(\frac{r_o}{r})}{(r-r_o)}\,dr_o \qquad (4.3\text{-}9)$$

and from equations (4.3-4) and (4.3-6)

$$w_t(r)_F = \frac{1}{4\pi}\int_{R_b}^{R_p} \frac{d\Gamma(r_0)}{dr_0}\left[\frac{i_t(\frac{r_0}{r})}{(r-r_0)} + \frac{b}{r}\right] dr_0$$

$$= \frac{1}{4\pi}\left[\frac{b\Gamma(R_p)}{r} + \int_{R_b}^{R_p} \frac{d\Gamma(r_0)}{dr_0}\frac{i_t(\frac{r_0}{r})}{(r-r_0)} dr_0\right] \qquad (4.3\text{-}10)$$

If there is clearance between the blade tip and duct, then as discussed in sectio (IV.1), the circulation at the tip is zero, $\Gamma(R_p) = 0$, and the first term does not exist. For convenience both of these equations will be non-dimensionalized; let

$$x = \frac{r}{R_p} = \frac{\text{reference radius}}{\text{propeller radius}}$$

$$x_o = \frac{r_o}{R_p} = \frac{\text{radius at which vortex is shed}}{\text{propeller radius}}$$

$$x_h = \frac{R_h}{R_p} = \frac{\text{hub radius}}{\text{propeller radius}}$$

and for the circulation

$$G_s = \frac{\Gamma}{2\pi R_p V_s} = (1 - W_x)G = \frac{(1 - W_x)\Gamma}{2\pi R_p W_o(x)} \qquad (4.3\text{-}11)$$

If the free-stream velocity is constant, then the free-stream velocity w_o is used for nondimensionalizing instead of the ship speed V_s.

Introducing this notation into equations (4.3-9) and (4.3-10) the induced velocities are given in nondimensional form.

$$\frac{w_a(x)_p}{V_s} = \frac{1}{2}\int_{x_h}^{1} \frac{dG_s}{dx_o}\,\frac{1}{(x-x_o)}\,i_a\,dx_o \tag{4.3-12}$$

and

$$\frac{w_t(x)_p}{V_s} = \frac{1}{2}\left[\frac{b}{x}\,G_s(1) + \int_{x_h}^{1}\frac{dG_s}{dx_o}\,\frac{1}{(x-x_o)}\,i_t\,dx_o\right]$$

or

$$\frac{w_t(x)_p}{V_s} - \frac{b\,G_s(1)}{2x} = \frac{1}{2}\int_{x_h}^{1}\frac{dG_s}{dx_o}\,\frac{1}{(x-x_o)}\,i_t\,dx_o \tag{4.3-13}$$

The integrand of the integrals in these equations are singular at $x = x_o$. Lerbs[33] has discussed both these equations except that the equations were derived for a propeller without the duct so that $G_s(1) = 0$. This term causes no difficulty, however, if it is rewritten on the left-hand side as shown since Lerbs deals with the same integral.

IV.4 Thrust and Torque Developed by the Propeller

The thrust on the propeller follows from the law of Kutta-Joukowski, equation (2.7-4). For each element of each blade this law states

$$dT = \rho V \Gamma dr = \rho \Gamma(r)(\omega_o r - w_{td} - w_{tp})dr \qquad (4.4\text{-}1)$$

where

> $\omega_o = 2\pi(\text{rps})$ is e angular velocity of the propeller, w_{td} is the tangential velocity induced at the propeller blade by the duct and includes the tangential induced velocities from the ring vortices, ring sources and duct trailing vortex system. w_{tp} is the tangential induced velocity by the trailing vortex system of the propeller, equation (4.3-13).

For the thrust the velocity V is the total tangential velocity by the propeller lifting-line excluding self-induced velocities. It should be noted that by assumption in Section III the tangential induced velocities from the hub are zero and for a lifting line the self induced velocities ($w_t^{(3)}$) are zero. The total thrust is obtained by integrating this equation from the propeller hub to the tip and summing over the number of blades. Since the thrust of each blade is the same, the thrust of one blade is multiplied by the number of blades.

$$T_i = \rho b \int_{R_d}^{R_p} \Gamma(r)\left[\omega_0 r - w_{t_d}(r) - w_{t_p}(r)\right] dr$$

(4.4-2)

This equation is nondimensionalized using the same notation as for equation (4.3-11) with the added definition of the advance coefficient

$$\lambda_s = \frac{V_s}{\omega_0 R_p} \qquad (4.4\text{-}3)$$

and the thrust coefficient of the propeller is

$$\left(C_{T_{si}}\right)_p = \frac{T}{\frac{\rho}{2} R_p^2 \pi V_s^2} = 4b \int_{x_h}^{1} G_s(x)\left[\frac{x}{\lambda_s} - \left(\frac{w_{t_d}}{V_s} + \frac{w_{t_p}}{V_s}\right)\right] dx \qquad (4.4\text{-}4)$$

If the free stream velocity w_c is uniform over the radius then it is used for nondimensionalizing instead of the ship speed. The subscript "i" is used to denote that the thrust is the thrust in an inviscid fluid. The total thrust of the ducted propeller is given by adding this equation to equation (2.7-9).

The torque is also obtained using the law of Kutta-Joukowski. This law gives a tangential force at each radius which when multiplied by the radius and integrated over the blade length gives the torque per blade.

$$dQ = \rho \tilde{V} \Gamma(r) dr = \rho r \Gamma(r)\left[w_0(r) + w_{a_h} + w_{a_d} + w_{a_p}\right] dr$$

where

$w_o(r)$ is the free-stream velocity

w_{ah} is the axial velocity induced by the hub

w_{ad} is the axial velocity induced by the duct

w_{ap} is the axial velocity induced by the trailing vortex system of the propeller, equation (4.3-9)

Integrating this equation from the hub to the blade tip and multiplying by the number of blades gives the total torque of the propeller.

$$Q = b\rho \int_{R_b}^{R_p} r\Gamma(r)\left[W_o(r) + w_{ab} + w_{ah} + w_{ap}\right]dr \tag{4.4-5}$$

Nondimensionalizing as before but defining a power coefficient as

$$C_{psi} = \frac{\omega_o Q}{\frac{\rho}{2} R_p^2 \pi \bar{V}_s^3}$$

the nondimensionalized form of this equation is

$$C_{psi} = \frac{4b}{\lambda_s}\int_{\chi_h}^{1} \chi G_s(\chi)\left[(1 - w_x) + \frac{w_a(r)}{V_s}_h + \frac{w_a(x)}{V_s}_d + \frac{w_a(r)}{V_s}_p\right] d\chi \tag{4.4-6}$$

where $(1-w_x)$ is the wake.

The ideal efficiency (inviscid) is given by the ratio of the total thrust to the power, i.e.

$$\eta_i = \frac{(C_{Tsi})_p + (C_{Tsi})_d}{C_{psi}} = \frac{(C_{Tsi})_d}{C_{psi}} + \eta_{ip} \tag{4.4-7}$$

where $(C_{tsi})_d$ is the duct thrust given by equation (2.7-9)

$$\eta_{ip} = \frac{(C_{tsi})_p}{C_{psi}} = \text{ideal efficiency of the propeller}$$

IV.5 The Integral Equation for the Circulation

The free helical vortices lie on stream surfaces and are assumed to have the pitch of the resultant flow angle at the lifting line. The velocities at a radius x are shown in the following velocity diagram.

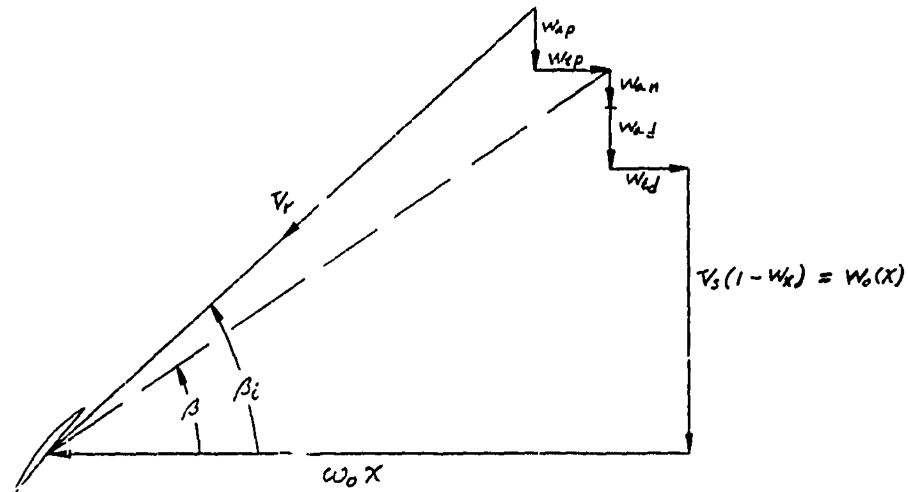

Figure 1. Velocity diagram of the ducted propeller

The helical vortex is shed at the angle β_i which is commonly called the hydrodynamic pitch angle. The angle β is the propeller advance angle.

From this velocity diagram the following equation is obtained for $\tan\beta_i$.

$$\tan\beta_i = \frac{W_o(x) + W_{ad} + W_{ah} + W_{ap}}{\omega_o x - W_{td} - W_{tp}} \tag{4.5-1}$$

$$= \frac{(1 - w_x) + \frac{W_a}{V_s}h + \frac{W_a}{V_s}d + \frac{W_a}{V_s}p}{\frac{x}{\lambda_s} - \frac{W_t}{V_s}d - \frac{W_t}{V_s}p}$$

Substituting in for $\frac{W_t}{V_s}p$ and $\frac{W_a}{V_s}p$, equations (4.3-12) and 4.3-13) this equation is obtained in terms of the unknown circulation

$$\tan\beta_i = \frac{(1-w_x) + \frac{W_a}{V_s}d + \frac{W_a}{V_s}h + \frac{1}{2}\int_{x_h}^{1}\frac{dG_s}{dx_o}\frac{1}{(x-x_o)}i_a\,dx_o}{\frac{x}{\lambda_s} - \frac{W_t}{V_s}d - \frac{1}{2}\frac{b}{x}G_s(1) - \frac{1}{2}\int_{x_d}^{1}\frac{dG_s}{dx_o}\frac{1}{(x-x_o)}i_t\,dx_o}$$

or

$$\frac{1}{2}\int_{x_h}^{1}\frac{dG_s}{dx_o}\frac{1}{(x-x_o)}\left[i_t\tan\beta_i + i_a\right]dx_o = \left[\frac{x}{\lambda_s} - \frac{W_t d}{V_s} - \frac{bG_s(1)}{2x}\right]\tan\beta_i$$

$$-(1-w_x) - \frac{W_a}{V_s}h - \frac{W_a d}{V_s} \qquad (4.5\text{-}2)$$

with boundary conditions

$$G_s(x_h) = 0 \quad and \quad G_s(1)\begin{cases} 0 \text{ if } R_p \neq R_d \\ A \text{ if } R_p = R_d \end{cases}$$

This last equation represents an integro-differential equation for the unknown circulation distribution which will give the desired thrust. Lerbs[33] gives a numerical method for solving this equation so only a few remarks will be made concerning it. The left hand side of this equation is the same as Lerbs discussed but the right hand side contains velocities induced by the hub and duct and, in addition, the circulation at the tip, if it is not zero. Since the right hand side is assumed known in either the free-running or ducted propeller case, this does not affect the solution method.

In a strict sense the induced velocities of the duct and hub at the propeller depend on the propeller circulation but because of the complexity of the problem these velocities must be assumed known. To consider that they are dependent on the propeller circulation in equation (4.5-2) implies that the circulation distribution representing the nozzle, the source distribution representing the hub, and the propeller blade circulation must be determined simultaneously. This of course is not possible and it is necessary to resort to a method of iteration. For instance, the duct and hub problem could be solved separately without the propeller and then using the resulting induced velocities the propeller problem solved. This process is then repeated, using each time the last derived induced velocities, until satisfactory convergence is obtained.

Equation (4.5-2) is in a general form as the free-stream velocity may vary radially (wake-adapted propellers) and the circulation distribution need not be optimum. In addition this equation applies to moderately loaded propellers* as well as, of course, to lightly loaded propellers. These various cases are discussed by Lerbs.[33]

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* The difference between a moderately loaded and a lightly loaded propeller is that for a moderately loaded propeller the velocities induced by the helical vortices are included in the calculation of the angle at which these vortices are shed while for a lightly loaded propeller their effect is ignored.

IV.6 The Optimum Circulation Distribution of the Propeller

For the free-running propeller a question arises as to what is the circulation distribution so that a propeller produces a given thrust with a minimum amount of power. This optimum circulation distribution is, of course, based on the lifting-line theory and an inviscid fluid. For the ducted propeller a similar question arises but the force on the duct itself enters the problem. The problem could also include determination of th. shape of the duct as well as the propeller circulation distribution. Dickmann and Weissinger[10] have considered this problem, the optimum shape of the duct, but for ducts of zero thickness and a simplified representation of the propeller.

The combined problem of optimum duct shape and optimum circulation distribution along the propeller blade is a formidable one since it is not possible, within the concepts of the theory developed here, to obtain the interference velocities in explicit form. This can be seen by referring to equation (4.5-2) in which it is necessary, in order to obtain a solution to assume that the induced velocities from the nozzle and hub are known and not functions of the circulation. For the same reason it is not feasible to take into consideration the total thrust of the ducted propeller system but to consider only the propeller thrust.

The problem, which can reasonably be solved, reduces to the determination of the circulation distribution on the

propeller blade so that the propeller produces a given thrust with minimum torque. This approach does not consider whether or not for this optimum circulation distribution the complete ducted system produces a given amount of thrust with minimum torque. For this reason the problem posed is somewhat academic and is discussed further only for the sake of completeness.

In the following analysis it will be assumed that the circulation at the blade ti is zero, $G_s(1) = 0$, and that the free-stream velocity is a constant, $w_o(x) = w_o$ or $(1 - w_x) = 1$. This essentially reduces the problem to a free-running, moderately loaded propeller in uniform flow with the addition of velocities induced by the duct and hub. Nondimensionalizing with the free-stream velocity, the thrust coefficient, equation (4.4-4), and power coefficient, equation (4.4-5) become

$$(C_{Ti})_p = \frac{T}{\frac{\rho}{2} R_p^2 \pi w_o^2} = 4b \int_{x_h}^{1} G(x) \left[\frac{x}{\lambda} - \left(\frac{w_{t}}{w_o}d + \frac{w_{t}}{w_o}p \right) \right] dx \tag{4.6-1}$$

and

$$C_{pi} = \frac{\omega_o Q}{\frac{\rho}{2} R_p^2 \pi w_o^3} = \frac{4b}{\lambda} \int_{x_h}^{1} x G(x) \left[1 + \frac{w_a(x)}{w_o}h + \frac{w_a(x)}{w_o}d + \frac{w_a(x)}{w_o}p \right] dx \tag{4.6-2}$$

where

$$\lambda = \frac{w_o}{\omega_o R_p}$$

$$G = \frac{\Gamma}{2\pi R_p w_o}$$

The propeller induced velocities, w_a and w_t, in these equations can be replaced by their values in terms of the circulation distribution, equations (4.3-12) and (4.3-13). After integration by parts the thrust and power coefficients can be written in terms of the derivative of the circulation distribution.

$$(C_{Ti})_p = -4b\int_{x_h}^{1}\left(\frac{(x^2-x_h^2)}{2\lambda} - \int_{x_h}^{x}\frac{w_{td}(x')}{w_0}dx' - \frac{1}{2}\int_{x_h}^{1}G'(x_0)\left[\int_{x_h}^{x}\frac{1}{(x'-x_0)}\,i_t\left(\frac{x_0}{x'}\right)dx'\right]dx_0\right)G'(x)dx \tag{4.6-3}$$

and

$$C_{pi} = -\frac{4b}{\lambda}\int_{x_h}^{1}\left(\frac{x^2-x_h^2}{2} + \int_{x_h}^{x}\left|\frac{w_a(x')}{w_0}_h - \frac{w_a(x')}{w_0}_d\right|x'dx' + \frac{1}{2}\int_{x_h}^{1}G'(x_0)\left[\int_{x_h}^{x}\frac{x'}{(x'-x_0)}\,i_t\left(\frac{x_0}{x'}\right)dx'\right]dx_0\right)G'(x)dx \tag{4.6-4}$$

In this form the thrust and power coefficients are functions of $G'(x)$ and not of both $G'(x)$ and $G(x)$. The problem is to find the circulation distribution $G(x)$ so that the power coefficient (C_{pi}) is a minimum while the thrust coefficient $(C_{Ti})_p$ remains unchanged. This is a problem in the calculus of variations.

A small variation is now taken of the slope of the circulation distribution, i.e. $G'(x) + \delta G'(x)$ represents the value of the slope of the circulation distribution in a small region surrounding the point x and at the end points the variation is zero, $\delta G'(x_h) = \delta G'(1) = 0$. If the circulation distribution is an optimum, then in this small region $C_{Ti}[G'(x) + \delta G'(x)]_p$ and $C_{Ti}[G'(x)]$ have the same value. If second order terms are ignored, the difference

$C_{Ti}[G'(x) + \delta G'(x)]_p - C_{Ti}[G'(x)]_p$ is obtained as follows:

$$(\delta C_{Ti})_p = C_{Ti}[G'(x)+\delta G'(x)]_p - C_{Ti}[G'(x)]_p = 2b\int_{x_h}^{1}\left(\int_{x_h}^{1}\delta G'(x_o)\left[\int_{x_h}^{x}\frac{1}{(x'-x_o)}i_t\left(\frac{x_o}{x'}\right)dx'\right]dx_o\right)G'(x)dx$$

$$-4b\int_{x_h}^{1}\left(\frac{(x^2-x_h^2)}{2\lambda}-\int_{x_h}^{x}\frac{w_{td}(x')}{w_o}dx' - \frac{1}{2}\int_{x_h}^{1}G'(x_o)\left[\int_{x_h}^{x}\frac{1}{(x'-x_o)}i_t\left(\frac{x_o}{x'}\right)dx'\right]dx_o\right)\delta G'(x)dx$$

$$= -4b\int_{x_h}^{1}\left(\frac{(x^2-x_h^2)}{2\lambda}-\int_{x_h}^{x}\frac{w_{td}(x')}{w_o}dx' - \frac{1}{2}\int_{x_h}^{1}G'(x_o)\left[\int_{x_h}^{x}\frac{1}{(x'-x_o)}i_t\left(\frac{x_o}{x'}\right)dx' + \int_{x_h}^{x_o}\frac{1}{(x'-x_o)}i_t\left(\frac{x}{x'}\right)dx'\right]dx_o\right)\delta G'(x)dx$$

$$= 0 \tag{4.6-5}$$

The variation $(\delta C_{Ti})_p$ is taken as zero since within the variation in $G'(x)$, $G_{Ti}[G'(x) + \delta G'(x)]_p - C_{Ti}[G'(x)]_p$ is taken as zero. A first order variation δC_{pi} may also be defined as

$$\delta C_{pi} = C_{pi}[G'(x)+\delta G'(x)]_p - C_{pi}[G'(x)]_p = -\frac{2b}{\lambda}\int_{x_h}^{1}\left(\delta G'(x)\left[\int_{x_h}^{x}\frac{x'}{(x'-x_o)}i_a\left(\frac{x_o}{x'}\right)dx'\right]dx_o\right)G'(x)dx$$

$$-\frac{4b}{\lambda}\int_{x_h}^{1}\left(\frac{(x^2-x_h^2)}{2}+\int_{x_h}^{x}\left[\frac{w_a(x')}{w_o}h + \frac{w_a(x')}{w_o}d\right]x'dx' - \frac{1}{2}\int_{x_h}^{1}G'(x_o)\left[\int_{x_h}^{x}\frac{x'}{(x'-x_o)}i_a\left(\frac{x_o}{x'}\right)dx'\right]dx_o\right)\delta G'(x)dx$$

$$= -\frac{4b}{\lambda}\int_{x_h}^{1}\left(\frac{(x^2-x_h^2)}{2}+\int_{x_h}^{x}\left[\frac{w_a(x')}{w_o}h + \frac{w_a(x')}{w_o}d\right]x'dx' + \frac{1}{2}\int_{x_h}^{1}G'(x_o)\left[\int_{x_h}^{x}\frac{x'}{(x'-x_o)}i_a\left(\frac{x_o}{x'}\right)dx' + \int_{x_h}^{x_o}\frac{x'}{(x'-x_o)}i_a\left(\frac{x}{x'}\right)dx'\right]dx_o\right)\delta G'(x)dx$$

$$= 0 \tag{4.6-6}$$

The variation δC_{pi} must be approximately zero for a sufficiently small variation in order for C_{pi} to be a minimum.

Actually C_{pi} by this analysis can be a maximum or a minimum and to show that C_{pi} for the optimum value of $G'(x)$ is a minimum it should be shown that $C_{pi}[\delta G'(x)] > 0$ for $\delta \neq 0$. This is not possible without knowing the form of $G'(x)$, however, it would be expected that C_{pi} is a minimum.

By equations (4.6-5) and (4.6-6) both δC_{ti_p} and δC_{pi} are zero for all $\delta G'(x)$ (δ sufficiently small) and this can be true if and only if the integrands of both these equations are proportional.. The equation that the circulation distribution must satisfy is then the following:

$$\frac{\dfrac{(x^2-x_h^2)}{2\lambda} - \displaystyle\int_{x_h}^{x} \frac{W_t(x')}{W_0}\Big|_d dx' - \frac{1}{2}\int_{x_h}^{1} G'(x_0)\left(\int_{x_h}^{x} \frac{1}{(x'-x_0)}\, i_t\!\left(\frac{x_0}{x'}\right) dx' + \int_{x_h}^{x_0} \frac{1}{(x'-x)}\, i_t\!\left(\frac{x}{x'}\right) dx'\right) dx_0}{\dfrac{(x^2-x_h^2)}{2} + \displaystyle\int_{x_h}^{x}\left[\frac{W_a(x')}{W_0}\Big|_h + \frac{W_a(x')}{W_0}\Big|_d\right] x'\,dx' + \frac{1}{2}\int_{x_h}^{x} G'(x_0)\left(\int_{x_h}^{x} \frac{x'}{(x'-x_0)}\, i_a\!\left(\frac{x_0}{x'}\right) dx' + \int_{x_h}^{x_0} \frac{x'}{(x-x')}\, i_a\!\left(\frac{x}{x'}\right) dx'\right) dx_0} = A$$

or

$$\int_{x_h}^{1} G'(x_0)\left(\int_{x_h}^{x} \frac{1}{(x'-x_0)}\left[i_t\!\left(\frac{x_0}{x'}\right) + A x' i_a\!\left(\frac{x_0}{x'}\right)\right] dx' + \int_{x_h}^{x_0} \frac{1}{(x'-x_0)}\left[i_t\!\left(\frac{x}{x'}\right) + A x' i_a\!\left(\frac{x}{x'}\right)\right] dx'\right) dx_0$$

$$= \left(\frac{1}{\lambda} - A\right)(x^2 - x_h^2) - 2\int_{x_h}^{x}\left(\frac{W_t}{W_0}(x')\Big|_d + \frac{W_a}{W_0}(x')\Big|_d + \frac{W_a}{W_0}(x')\Big|_h\right) dx' \qquad (4.6\text{-}8)$$

This is an integral equation of the first kind for the circulation distribution in terms of the constant A. As discussed in section III direct inversion of an integral of this type is usually not possible, however Landweber[31] has discussed an iteration procedure for this type of integral. The solution of this equation will not be discussed further. Once the form of the circulation is obtained then the value

of the parameter "A" can be calculated from the equation for the thrust coefficient equation (4.6-3), and the value will depend on the value of the thrust coefficient.

If the propeller is a free-running propeller, then

$\frac{w_t(x')_d}{w_o}$ and $\frac{w_a(x')_d}{w_o}$ are zero and this equation (4.6-8) reduces to that for the optimum circulation distribution for a moderately loaded propel r*. (The velocity induced by the hub is usually neglected.)

By interchanging the order of integration and then integrating by parts, equation (4.6-7) can be written in the following form:

$$\frac{\int_{x_h}^{x}\left(\frac{x'}{\lambda} - \frac{w_t(x')}{w_o}d - \frac{w_t(x')}{w_o}p\right)dx' + \frac{1}{2}\int_{x_h}^{1}\frac{G(x')}{(x'-x)}i_t\left(\frac{x}{x'}\right)dx'}{\int_{x_h}^{x}\left(1 + \frac{w_a(x')}{w_o}d + \frac{w_a(x')}{w_o}h + \frac{w_a(x')}{w_o}p\right)x'dx' - \frac{1}{2}\int_{x_h}^{1}\frac{x'G(x')}{(x'-x)}i_a\left(\frac{x}{x'}\right)dx'} = A$$

After differentiating and inverting, this equation becomes

$$\frac{\left[1 + \frac{w_a(x')}{w_o}d + \frac{w_a(x')}{w_o}h + \frac{w_a(x')}{w_o}p\right]x' - \frac{1}{2}\frac{\partial}{\partial x}\int_{x_h}^{1}\frac{x'G(x')}{(x'-x)}i_a\left(\frac{x}{x'}\right)dx'}{\left[\frac{x'}{\lambda} - \frac{w_t(x')}{w_o}d - \frac{w_t(x')}{w_o}p\right] - \frac{1}{2}\frac{\partial}{\partial x}\int_{x_h}^{1}\frac{G(x')}{(x'-x)}i_t\left(\frac{x}{x'}\right)dx'} = A$$

If the integrals are zero in this equation, then from Figure 7 it can be seen that this implies $x \cdot \tan\beta_i$ = constant. This is mentioned in light of Betz's theorem which states that

* This particular form was obtained by Prof. J.V. Wehausen in his unpublished class notes for "Hydrodynamics of Ships".

for a free-running lightly loaded propeller the circulation distribution is optimum if the angle β_i satisfies $x \tan \beta_i$ = constant. It is not obvious that equation (4.6-9) will result in this theorem even if the propeller is assumed to be free-running and lightly loaded.

V. Interaction Effects

When discussing the duct and hub in Sections II and III, the form of the propeller induced velocities was not included because they had not yet been derived. This section then will deal mainly with the effect of the propeller induced velocities on the duct, however, the equations for the duct and propeller induced velocities at the hub will be given in a more explicit form than in Section III.

V.1 The Duct Trailing Vortex System

As stated in Section II, when an annular airfoil is subjected to a radial velocity which is dependent on the angular position, a trailing vortex of strength $\frac{1}{R_d}\frac{\partial\gamma}{\partial\varphi}$ is shed from each point (φ,z) on the duct. This trailing vortex system is shed at an angle equal to the flow angle and follows a stream line in the rotating coordinate system. This implies, as in case of the propeller, that the induced velocities from all the components in the flow field have an effect on the trailing vortex system. This represents a problem considerably more difficult than the propeller problem since the helical vortices are shed from all over the duct rather than along a line. To obtain the equation in a form which is amenable to solution and yet which, it is felt, represents the flow field rather well, it will be assumed that the helical vortices are all shed at the advance angle of the duct. This is the angle

given by the following equation

$$\tan\beta_d = \frac{w_0(R_d)}{\omega_0 R_d} = \frac{(1-W_d)V_s}{\omega_0 R_d} \tag{5.1-1}$$

It will further be assumed that these vortices maintain a constant pitch angle and form a cylindrical vortex sheet of diameter R_d extending from the duct to minus infinity in the axial direction.

The velocity induced by a single helical vortex line is derived in Appendix D. The velocity components induced by the cylindrical vortex sheet shed from the duct are obtained by integrating equation (D-7), (D-8) and (D-9) over the surface of the duct. Each vortex shed from the duct has a radius R_d, pitch angle β_d and strength $\frac{1}{R_d}\frac{\partial\gamma}{\partial\varphi}$ so these components can be written in the following nondimensionalized form

$$\left[\frac{W_a}{V_s}\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\partial\gamma}{\partial\varphi'}(\varphi',z')\left(\int_{-\infty}^{0}\left[\frac{1-\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi'-\alpha)}{R^3}\right]d\alpha\right)d\varphi'dz' \tag{5.1-2}$$

$$\left[\frac{W_r}{V_s}\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\partial\gamma}{\partial\varphi'}(\varphi',z')\left(\int_{-\infty}^{0}\left[\frac{[2h(\bar{z}-z')-\alpha\tan\beta_d]\cos(\varphi-\varphi'-\alpha)-\tan\beta_d\sin(\varphi-\varphi'-\alpha)}{R^3}\right]d\alpha\right)d\varphi'dz' \tag{5.1-3}$$

$$\left[\frac{W_t}{V_s}\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\partial\gamma}{\partial\varphi'}(\varphi',z')\left(\int_{-\infty}^{0}\left[\frac{[-2h(\bar{z}-z')+\alpha\tan\beta_d]\sin(\varphi-\varphi'-\alpha)+\left[\left(\frac{x}{x_d}\right)-\cos(\varphi-\varphi'-\alpha)\right]\tan\beta_d}{R^3}\right]d\alpha\right)d\varphi'dz' \tag{5.1-4}$$

where

$$R^2 = 1+\left(\frac{x}{x_d}\right)^2-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi_p-\alpha)+[2h(\bar{z}-z')-\alpha\tan\beta_d]^2$$

At the propeller reference blade z=0 (i.e. $\bar{z}=-a_t$) and φ =0, so the above equations are considerably simplified at that point. In order to obtain the circulation distribution on the duct it is necessary to have the radial velocity induced on the duct by the trailing vortex system. This is given by letting $x=x_d$ in equation (5.1-3).

$$\left[\frac{w_r}{V_s}(x_d,\varphi,\bar{z})\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\partial\gamma(\varphi',z')}{\partial\varphi'}\left(\int\left[\frac{[2h(\bar{z}-z')-\alpha\tan\beta_d]\cos(\varphi-\varphi'-\alpha)-\tan\beta_d\sin(\varphi-\varphi'-\alpha)}{\{2[1-\cos(\varphi-\varphi'-\alpha)]+[2h(\bar{z}-z')-\alpha\tan\beta_d]^2\}^{3/2}}\right]d\alpha\right)d\varphi' dz' \tag{5.1-5}$$

The uniform convergence of the infinite integral in this equation is discussed in Appendix D.

V.2 The Radial Velocities Induced at the Duct by the Propeller and Hub

The radial velocity induced on the duct surface by the propeller is given by integrating equations (4.3-3) and (4.3-5) along the radius and summing the two equations. The circulation denoted by $\bar{\Gamma}$ for a single vortex when referred to the circulation at the lifting line is $\frac{d\Gamma(r_0)}{dr_0}$. Substituting this for $\bar{\Gamma}$ in equation (4.3-3) and integrating from the hub to the tip, the trailing vortex system induces the following radial velodity on the duct.

$$\left[W_r^{(1)}\right]_p = \frac{1}{4\pi}\int_{r_h}^{R_p}\frac{d\Gamma}{dr_0}r_0\sum_{p=1}^{b}\int_{-\infty}^{0}\left\{\frac{[\xi-\alpha r_0\tan\beta_i]\cos(\varphi-\varphi_p-\alpha)-r_0\tan\beta_i\sin(\varphi-\varphi_p-\alpha)}{[R_d^2+r_0^2+(\xi-r_0\alpha\tan\beta_i)^2-2R_d r_0\cos(\varphi-\varphi_p-\alpha)]^{3/2}}\right\}d\alpha\, dr_0$$

If this equation is nondimensionalized as previously, equation (4.3-10), then it can be written in the following form.

$$\left[\frac{W_r^{(1)}}{V_s}\right]_p = \frac{1}{2x_d}\int_{x_h}^{1}G_s'(x_0)\,\bar{i}_r^{(1)}\left(\frac{x_0}{x_d},\varphi,z\right)dx_0 \tag{5.2-1}$$

$$\bar{i}_r^{(1)}\left(\frac{x_0}{x_d},\varphi\right) = \left(\frac{x_0}{x_d}\right)\left(\frac{1}{x_d}\right)\sum_{p=1}^{b}\int_{-\infty}^{0}\left\{\frac{[2hz-\alpha(\frac{x_0}{x_d})\tan\beta_i]\cos(\varphi-\varphi_p-\alpha)-(\frac{x_0}{x_d})\tan\beta_i\sin(\varphi-\varphi_p-\alpha)}{[1+(\frac{x_0}{x_d})^2+(2hz-\alpha\frac{x_0}{x_d}\tan\beta_i)^2-2(\frac{x_0}{x_d})\cos(\varphi-\varphi_p-\alpha)]^{3/2}}\right\}d\alpha \tag{5.2-2}$$

As discussed in Appendix D the integrand has a finite jump discontinuity when the point P(x, z) lies on the helix and $z \neq 0$ or $\varphi \neq \varphi_p$.

This can only occur in the foregoing equation when the duct has the same diameter as the propeller ($x_d=1=x_o$) and $2hz=(\varphi-\varphi_p)\tan\beta_i$. At this point the integrand has a finite jump since

$$\lim_{\alpha\to(\varphi-\varphi_p)}\Big(\qquad\Big) = \mp\frac{\tan\beta_i}{3(1+\tan^2\beta_i)^{1/2}}\,,\quad z\neq 0 \ \text{ or } \ \varphi\neq\varphi_p$$

If $z=0$ and $\varphi=\varphi_p$, then the integrand is singular as discussed in Appendix D. For calculating purposes it is probably best to nondimensionalize the axial component with the propeller radius, since this equation is only dependent on the duct chord through the non-dimensionalization of the axial component. In this form then equation (5.2-2) is

$$\bar{i}_r^{(1)}\left(\frac{x_o}{x_d},\varphi\right)=\left(\frac{x_o}{x_d}\right)\left(\frac{1}{x_d}\right)\sum_{p=1}^{b}\int_{-\infty}^{0}\left(\frac{\left[\left(\frac{z_p}{x_d}\right)-\alpha\left(\frac{x_o}{x_d}\right)\tan\beta_i\right]\cos(\varphi-\varphi_p-\alpha)-\left(\frac{x_o}{x_d}\right)\tan\beta_i\sin(\varphi-\varphi_p-\alpha)}{\left[1+\left(\frac{x_o}{x_d}\right)^2+\left(\frac{z_p}{x_d}-\alpha\frac{x_o}{x_d}\tan\beta_i\right)^2-2\left(\frac{x_o}{x_d}\right)\cos(\varphi-\varphi_p-\alpha)\right]^{3/2}}\right)d\alpha \tag{5.2-3}$$

where z_p is the axial coordinate nondimensionalized by the propeller radius R_p.

This equation is independent of the duct chord and once tabulated on the basis of z_p the values can easily be changed to those for the axial coordinate z by $z=\frac{z_p}{2h\,x_d}$. It should be noted that this factor $\bar{i}_r$ is not the normal induction factor[33] i_r which is $i_r=(1-\frac{x_o}{x})\,\bar{i}_r$. If the velocity induced at the hub by trailing vortex system x_h is introduced in the foregoing equations in place of x_d, then this equation gives the radial velocity at the hub.

The radial velocity induced by the line vortex itself is obtained from equation (4.2-10). In nondimensionalized form

this velocity is

$$\left[\frac{w_r^{(3)}}{V_s}\right]_p = -\frac{1}{2}\int_{x_h}^{1}\frac{G_s'(x_o)}{x}\sum_{p=1}^{b}\left(\frac{\left(\frac{z_p}{x}\right)\sin(\varphi-\varphi_p)}{\left[\left(\frac{z_p}{x}\right)^2+\sin^2(\varphi-\varphi_p)\right]}\left[\frac{\left(\frac{x_o}{x}\right)-\cos(\varphi-\varphi_p)}{\sqrt{\left(\frac{z_p}{x}\right)^2+1+\left(\frac{x_o}{x}\right)^2-2\left(\frac{x_o}{x}\right)\cos(\varphi-\varphi_p)}}+\frac{\cos(\varphi-\varphi_p)}{\sqrt{\left(\frac{z_p}{x}\right)^2+1}}\right]\right)dx_o \tag{5.2-4}$$

For $z_p=o$ or $\varphi=\varphi_p \pm n$,(n=0,1,2...)

$$\left[\frac{w_r^{(3)}}{V_s}\right]_p = 0$$

This equation is also written independently of the duct chord by nondimensionalizing with the propeller radius, and as before $z_p=2hx_dz$. This equation is in a general form and can be applied at the propeller hub or the duct by using either the hub radius x_h or duct radius x_d instead of x in the above equation. If a factor $\bar{i}_r^{(3)}$ is defined as for the free vortex sheets, then the above equation can be written in a more simplified form

$$\left[\frac{w_r^{(3)}}{V_s}\right]_p = -\frac{1}{2}\int_{x_h}^{1} G_s'(x_o)\,\bar{i}_r^{(3)}\left(\frac{x_o}{x},\varphi,z_p\right)dx_o \tag{5.2-5}$$

where

$$\bar{i}_r^{(3)}\left(\frac{x_o}{x},\varphi,z_p\right) = \frac{1}{x}\sum_{p=1}^{b}\left(\frac{\left(\frac{z_p}{x}\right)\sin(\varphi-\varphi_p)}{\left[\left(\frac{z_p}{x}\right)^2+\sin^2(\varphi-\varphi_p)\right]}\left[\frac{\left(\frac{x_o}{x}\right)-\cos(\varphi-\varphi_p)}{\sqrt{\left(\frac{z_p}{x}\right)^2+1+\left(\frac{x_o}{x}\right)^2-2\left(\frac{x_o}{x}\right)\cos(\varphi-\varphi_p)}}+\frac{\cos(\varphi-\varphi_p)}{\sqrt{\left(\frac{z_p}{x}\right)^2+1}}\right]\right) \tag{5.2-6}$$

And combining this equation with equation (5.2-5), the total radial induced velocity by the propeller is given as

$$\left[\frac{w_r}{V_s}(x,\varphi,z)\right]_p = \frac{1}{2}\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r^{(1)}\left(\frac{x_o}{x},\varphi,z\right) - \bar{i}_r^{(3)}\left(\frac{x_o}{x},\varphi,z\right)\right]dx_o \tag{5.2-7}$$

or at the duct $x=x_d$

$$\left[\frac{w_r}{V_s}(x_d,\varphi,z)\right] = \frac{1}{2}\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r^{(1)}\left(\frac{x_o}{x_d},\varphi,z\right) - \bar{i}_r^{(3)}\left(\frac{x_o}{x_d},\varphi,z\right)\right]dx_o$$

$$= \frac{1}{2}\int_{x_h}^{1} G_s'(x_o)(\bar{i}_r)_p\, dx_o \tag{5.2-8}$$

For any ducted system the circulation distribution G_s is a function of the duct radius x_d the pitch angle β_i, the number of blades b and the duct and hub shape. It is written here as only a function of the radius x_o since for any one configuration it can only be a function of the radius. It should be noted that the factor $(\bar{i}_r)_p$ contains both odd and even terms.

The radial velocity induced at the duct by the hub is obtained from equation (3-10) by letting $x=x_d$

$$\left[\frac{w_r}{V_s}(x_d, z)\right]_h = \frac{h}{x_d}\int_{b_1}^{b_2}\frac{q(z')\,dz'}{[4h^2(z-z')^2+1]^{3/2}} \tag{5.2-9}$$

The function q(z') is the source strength of the line source representing the hub and is given by equation (3-14).

V.3 Integral Equation for the Circulation Distribution of the Duct Ring Vortices.

It was shown in Section II.3 that the ring source distribution was a function of the annular airfoil thickness only. Using this fact and substituting into equation (2.3-3) the radial velocities induced at the duct by the duct trailing vortex sheet, propeller, and hub, equations (5.1-5, 5.2-7 and 5.2-9), the integral equation for the vortex distribution is obtained as

$$U(\varphi,\bar{z}) = h\int_0^1\int_{-\pi}^{\pi}\frac{2h(\bar{z}-z')\cos(\varphi-\varphi')\,\gamma(\varphi',z')}{[4h^2(\bar{z}-z')^2+2\{1-\cos(\varphi-\varphi')\}]^{3/2}}\,d\varphi'\,dz'$$

$$+h\int_0^1\int_{-\pi}^{\pi}\frac{\partial\gamma(\varphi',z')}{\partial\varphi'}\left(\int_{-\infty}^{0}\left[\frac{[2h(\bar{z}-z')-\alpha\tan\beta_d]\cos(\varphi-\varphi'-\alpha)-\tan\beta_d\sin(\varphi-\varphi'-\alpha)}{\{2[1-\cos(\varphi-\varphi'-\alpha)]+[2h(\bar{z}-z')-\alpha\tan\beta_d]^2\}^{3/2}}\right]d\alpha\right)d\varphi'\,dz'$$

where

$$U(\varphi,\bar{z}) = -(1-W_{xd})\left(2\pi[c'(\bar{z})+\tan\alpha]+2h\int_0^1 s'(z')\,k\,[K(k)-E(k)]\,dz'\right)$$

$$-\pi\int_{x_h}^{1}G_s'(x_o)\left[\bar{\iota}_{r}\left(\frac{x_o}{x_d},\varphi,\bar{z}\right)\right]_p dx_o \;-\; 2\pi\frac{h}{x_d}\int_{b_1}^{b_2}\frac{q_h(z')\,dz'}{[4h^2(\bar{z}-z')^2+1]^{3/2}}$$

(5.3-2)

This equation is a singular integro-differential equation in two-dimensions for the circulation distribution. The part of the equation to the left of the equal sign is assumed known, i.e. the duct shape and the radial velocities induced by the propeller and hub. It should be mentioned that it would be expected with the normal configuration the radial velocity induced by the hub at the duct would be negligible.

The integral equation can be reduced to a one-dimensional one if it is assumed that the ring vortex strength can be expanded in a Fourier Series in φ ,[19] i.e.

$$\gamma(\varphi,z) = \sum_{n=0}^{\infty} g_n(z)\cos n\varphi + \sum_{n=1}^{\infty} h_n(z)\sin n\varphi \tag{5.3-3}$$

$$\left.\begin{aligned} g_0(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(\varphi,z)\,d\varphi \\ g_n(z) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \gamma(\varphi,z)\cos n\varphi\,d\varphi \\ h_n(z) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \gamma(\varphi,z)\sin n\varphi\,d\varphi \end{aligned}\right\} \tag{5.3-4}$$

For convenience the part of the integral equation denoted by U(φ ,z) will also be expanded in a Fourier series in φ . This involves no assumptions on the form of U(φ ,z) since all the functions in U(φ ,z) are continuous with respect to φ .

$$U(\varphi,z) = \sum_{n=0}^{\infty} u_n(z)\cos n\varphi + \sum_{n=0}^{\infty} v_n(z)\sin n\varphi \tag{5.3-5}$$

$$\left.\begin{aligned} u_0(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\varphi,z)\,d\varphi \\ u_n(z) &= \frac{1}{\pi}\int_{-\pi}^{\pi} U(\varphi,z)\cos n\varphi\,d\varphi \\ v_n(z) &= \frac{1}{\pi}\int_{-\pi}^{\pi} U(\varphi,z)\sin n\varphi\,d\varphi \end{aligned}\right\} \tag{5.3-6}$$

From equation (5.3-2) it is apparent that $u_n(z)$ and $v_n(z)$ for $n \geqq 1$ will contain terms involving only the propeller induced velocities. The Fourier coefficient $u_o(z)$ is a function of the duct shape, radial induced velocities from the hub and the average radial induced velocity from the propeller. It can immediately be written as

$$u_o(z) = -2(1-w_{x_d})\left\{\pi\left[c'(z)+\tan\alpha\right] + h\int_0^1 s'(z')\,k\left[K(k)-E(k)\right]dz'\right\}$$

$$-\frac{1}{2}\int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r\left(\tfrac{x_o}{x_d},\varphi,z\right)\right]_p dx_o\,d\varphi \;-\; 2\pi\frac{h}{x_d}\int_{b_1}^{b_2}\frac{q_h(z')\,dz'}{\left[4h^2(z-z')^2+1\right]^{3/2}}$$

$$= -\frac{1}{2}H(z) - 2\pi\frac{h}{x_d}\int_{b_1}^{b_2}\frac{q_h(z')\,dz'}{\left[4h^2(z-z')^2+1\right]^{3/2}} - \frac{1}{2}\int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r\left(\tfrac{x_o}{x_d},\varphi,z\right)\right]_p dx_o\,d\varphi \qquad (5.3\text{-}7)$$

The function H(z) is the same as that given by equation (2.4-3). Since the integrals in this equation are functions of hub shape and circulation distribution on the propeller lifting line and are assumed known for the duct problem they can be evaluated. The integral dependent on the hub shape is simple enough that it can easily be solved numerically, if necessary, and the integral involved the blade lifting line can be reduced to a more simple form for numerical solution. Consider the following integral

$$\bar{u}_o(z) = \int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r\left(\tfrac{x_o}{x_d},\varphi,z\right)\right]_p dx_o\,d\varphi = \int_{x_h}^{1} G_s'(x)\left[\int_{-\pi}^{\pi}\bar{i}_r\left(\tfrac{x_o}{x_d},\varphi,z\right)d\varphi\right]_p dx_o$$

$$= \int_{x_h}^{1} G_s'(x_o)\left[\int_{-\pi}^{\pi} \bar{\iota}_r^{(1)}\left(\frac{x_o}{x_d}, \varphi, z\right) d\varphi\right] dx_o - \int_{x_h}^{1} G_s'(x_o)\left[\int_{-\pi}^{\pi} \bar{\iota}_r^{(3)}\left(\frac{x_o}{x_d}, \varphi, z\right) d\varphi\right] dx_o \tag{5.3-8}$$

The integral involving the lifting line can easily be shown to be zero. From equation (5.2-5)

$$\int_{-\pi}^{\pi} \bar{\iota}_r^{(3)}\left(\frac{x_o}{x_d}, \varphi, z\right) d\varphi = \frac{1}{x_d}\sum_{p=1}^{b}\int_{-\pi}^{\pi}\left(\frac{\left(\frac{z\rho}{x_d}\right)\sin(\varphi-\varphi_p)}{\left[\left(\frac{z\rho}{x_d}\right)^2+\sin^2(\varphi-\varphi_p)\right]}\left[\frac{\left(\frac{x_o}{x_d}\right)-\cos(\varphi-\varphi_p)}{\sqrt{\left(\frac{z\rho}{x_d}\right)^2+1+\left(\frac{x_o}{x_d}\right)^2-2\left(\frac{x_o}{x_d}\right)\cos(\varphi-\varphi_p)}} - \frac{\cos(\varphi-\varphi_p)}{\sqrt{\left(\frac{z\rho}{x_d}\right)^2+1}}\right]\right) d\varphi$$

Letting $\theta = (\varphi - \varphi_p)$ then

$$\int_{-\pi}^{\pi} \bar{\iota}_r^{(3)}\left(\frac{x_o}{x_d}, \varphi, z\right) d\varphi = \frac{b}{x_d}\int_{-\pi}^{\pi}\left(\frac{\left(\frac{z\rho}{x_d}\right)\sin\theta}{\left[\left(\frac{z\rho}{x_d}\right)^2+\sin^2\theta\right]}\left[\frac{\left(\frac{x_o}{x_d}\right)-\cos\theta}{\sqrt{\left(\frac{z\rho}{x_d}\right)^2+1+\left(\frac{x_o}{x_d}\right)^2-2\left(\frac{x_o}{x_d}\right)+4\sin^2\frac{1}{2}\theta}} - \frac{\cos\theta}{\sqrt{\left(\frac{z\rho}{x_d}\right)^2+1}}\right]\right) d\varphi$$

$$= 0 \tag{5.3-9}$$

The integral involving the helical vortex sheet can be reduced to a function of elliptic integrals. From equation (5.2-2)

$$\int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o)\,\bar{\iota}_r^{(1)}\,dx_o\,d\varphi \tag{5.3-10}$$

$$= \int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o)\left(\frac{x_o}{x_d}\right)\left(\frac{1}{x_d}\right)\sum_{p=1}^{b}\int_{-\infty}^{0}\left(\frac{\left[2hz-\alpha\left(\frac{x_o}{x_d}\right)\tan\beta_i\right]\cos(\varphi-\varphi_p-\alpha)-\left(\frac{x_o}{x_d}\right)\tan\beta_i\sin(\varphi-\varphi_p-\alpha)}{\left[1+\left(\frac{x_o}{x_d}\right)^2+\left(2hz-\alpha\frac{x_o}{x_d}\tan\beta_i\right)^2-2\left(\frac{x_o}{x_d}\right)\cos(\varphi-\varphi_p-\alpha)\right]^{3/2}}\right) d\alpha\, dx_o\, d\varphi$$

The infinite integral in this equation is uniformly convergent with respect to $\varphi-\varphi_p$ and x_o if $z\neq 0$ and consequently the order of integration can be interchanged (see Appendices D and E). By changing the order of integration, the integration with respect to α and φ, can be carried out.

$$\bar{u}_o(z) = \int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o)\,\bar{\iota}_r^{(1)}\,dx_o\,d\varphi = \int_{x_h}^{1} G_s'(x_o)\left[\int_{-\pi}^{\pi}\bar{\iota}_r^{(1)}\,d\varphi\right] dx_o$$

$$= \int_{x_h}^{1} G_s'(x_o)\left(\frac{x_o}{x_d}\right)\left(\frac{1}{x_d}\right)\sum_{p=1}^{b}\int_{-\pi}^{\pi}\int_{-\infty}^{0}\left(\frac{\left[2hz-\alpha\left(\frac{x_o}{x_d}\right)\tan\beta_i\right]\cos(\varphi-\varphi_p-\alpha)-\left(\frac{x_o}{x_d}\right)\tan\beta_i\sin(\varphi-\varphi_p-\alpha)}{\left[1+\left(\frac{x_o}{x_d}\right)^2+\left(2hz-\alpha\frac{x_o}{x_d}\tan\beta_i\right)^2-2\left(\frac{x_o}{x_d}\right)\cos(\varphi-\varphi_p-\alpha)\right]^{3/2}}\right) d\alpha\, d\varphi\, dx_o$$

$$= b\int_{x_h}^{1} G_s'(x_o)\left(\frac{x_o}{x_d}\right)\left(\frac{1}{x_d}\right)\left[\int_{-\pi}^{\pi}\int_{0}^{\infty}\left(\frac{[2hz+\bar{\alpha}(\frac{x_o}{x_d})\tan\beta_i]\cos(\bar{\varphi}+\bar{\alpha}) - (\frac{x_o}{x_d})\tan\beta_i\sin(\bar{\varphi}+\bar{\alpha})}{[1+(\frac{x_o}{x_d})^2+(2hz+\bar{\alpha}\frac{x_o}{x_d}\tan\beta_i)^2-2(\frac{x_o}{x_d})\cos(\bar{\varphi}+\bar{\alpha})]^{3/2}}\right)d\bar{\alpha}\,d\bar{\varphi}\right]dx_o$$

(5.3-11)

The change in variable $\alpha = -\bar{\alpha}$ has been made in this equation and also the angle of the blade does not appear since it makes no difference what complete cycle of a periodic function is integrated. The integral in the brackets is discussed in Appendix E. The reduced form is obtained by taking the value of equation (E-18) at n=0. This gives

$$\left[\int_{-\pi}^{\pi}\int_{0}^{\infty}\left(\quad\right)d\alpha\,d\bar{\varphi}\right] = \frac{4}{(\frac{x_o}{x_d})^2\tan^2\beta_i}\int_{0}^{\frac{\pi}{2}}\frac{\cos 2\phi}{\sqrt{(\varsigma^2+\sigma^2)}}\,d\phi \qquad (5.3\text{-}12)$$

where

$$\varsigma = \frac{2hz}{\tan\beta_i}\left(\frac{x_d}{x_o}\right)$$

$$\sigma^2 = \frac{1}{\tan^2\beta_i}\left[\left(1+\frac{x_o}{x_d}\right)^2 - 4\left(\frac{x_o}{x_d}\right)\cos^2\phi\right]\left(\frac{x_d}{x_o}\right)^2$$

After integrating part of the integrand, this integral has the following form.

$$\left[\int_{-\pi}^{\pi}\int_{0}^{\infty}\left(\quad\right)d\alpha\,d\bar{\varphi}\right] = \frac{2}{k\tan\beta_i}\left(\frac{x_d}{x_o}\right)^{3/2}\left[(2-k^2)K(k) - 2E(k)\right]$$

$$= \frac{2}{\tan\beta_i}\, i_{r0}\left(\frac{x_o}{x_d}, z\right) \qquad (5.3\text{-}13)$$

where

$$k^2 = \frac{4\left(\frac{x_o}{x_d}\right)}{\left(1+\frac{x_o}{x_d}\right)^2 + 4h^2 z^2}$$

The double integral given by equation (5.3-13) does not exist for k=1, i.e. $x_o = x_d = 1$ and z=0. The asymptotic behavior is such that the integral is infinite as $\ln \infty$ at this point which is also the value obtained after interchanging the order of integration. When k=0 the value of the double integral is zero since[36]

$$\lim_{k \to 0} \left(\frac{K-E}{k^2} \right) = 0$$

The singularity at k=1 of the right-hand side of equation (5.3-13) is due to the elliptic integral of the second kind K(k), however this causes little difficulty since this singularity occurs under the integral with respect to x_o and a logarithmic singularity integrates out. If equation (5.3-13) is introduced into equation (5.3-8) and a change of variable is made the logarithmic singularity can be removed.

Assuming $x_d=1$ and z=0 then let $(1-\frac{x_o}{x_d}) = t^3$

$$\frac{2b}{x_d} \int_{x_h}^{1} \frac{G_1'(x_o)}{\tan\beta_i} \left(\frac{x_o}{x_d}\right) \bar{i}_{r_o}\left(\frac{x_o}{x_d}, z\right) dx_o$$

$$= 6b \int_{0}^{\sqrt[3]{1-x_h}} \frac{G_1'(t)}{\tan\beta_i} (1-t^3)\, \bar{i}_{r_o}(1-t^3, 0)\, t^2 dt$$

$$= 6b \int_0^{\sqrt[3]{1-x_h}} \frac{G_s'(t)}{\tan\beta_i} (1-t^3)^{-\frac{1}{2}} \frac{t^2}{k_2} \left[(2-k_2^2) K(k_2) - 2E(k_2) \right] dt \tag{5.3-14}$$

where

$$k_2 = \frac{4(1-t^3)}{(2-t^3)^2}$$

The elliptic integral $K(k_2)$ has a logarithmic singularity at $k_2=1$, however

$$\lim_{t\to o} t^2 K(k_2) = 0$$

and the integrand of the above integral is no longer singular. Using equation (5.3-13) the function $\bar{u}_o(z)$ becomes

$$\bar{u}_o(z) = \int_{-\pi}^{\pi}\int_{x_h}^{1} G_s'(x_o) i_r^{(1)} dx_o d\varphi = \frac{2b}{x_d}\int_{x_h}^{1} \frac{G_s'(x_o)}{\tan\beta_i}\left(\frac{x_o}{x_d}\right) \bar{i}_{r_o}\left(\frac{x_o}{x_d}, z\right) dx_o \tag{5.3-15}$$

A similar development holds for the average radial propeller induced velocity on the hub. In this case x_h is substituted for x_d in equation (5.3-15) and the singularity at $x_o=x_h$, $z=0$ is removed by making the substitution $\left(\frac{x_o}{x_h}-1\right) = t^3$.

Substituting equation (5.3-15) into equation (5.3-7), $u_o(z)$ has the following form

$$u_o(z) = -2H(z) - 2\pi \frac{h}{x_d} \int_{b_1}^{b_2} \frac{g_h(z')dz'}{[4h^2(z-z')^2+1]^{3/2}} - \frac{2b}{x_d}\int_{x_h}^{1} \frac{G_s'(x_o)}{\tan\beta_i}\left(\frac{x_o}{x_d}\right)\bar{i}_{r_o} dx_o \tag{5.3-16}$$

$$= -2\left[H(z) + H_h(z) + H_p(z)\right]$$

The factor $\bar{i}_{ro}$ is given by equation (5.3-12) however when $z=0$ and $x_d=1$, equation (5.3-14) should be used for this equation, i.e.

$$u_o(0) = -2H(0) - 2\pi h \int_{b_1}^{b_2} \frac{q_h(z')dz'}{[4h^2 z'^2+1]^{3/2}} - 3b \int_0^{\sqrt[3]{1-x_h}} \frac{G_i'(t)}{\tan\beta_i}(1-t^3)^{-\frac{1}{2}} \frac{t^2}{k_2^2}\left[(2-k_2^2)K(k_2) - 2E(k_2)\right]dt \tag{5.3-17}$$

where

$$k_2^{\,2} = \frac{4(1-t^3)}{(2-t^3)^2}$$

It is obvious by the form of $U(\varphi,z)$, equation (5.3-2), and the above form of $u_o(z)$ that the Fourier coefficients of higher order, $g_n(z)$ and $h_n(z)$ for $n \geqq 1$, are functions only of the propeller induced velocity. From equation (5.2-8) and equation (5.3-6) the Fourier coefficients $u_n(z)$ and $v_n(z)$ then follow as

$$u_n(z) = -\frac{1}{2}\int_{-\pi}^{\pi}(\cos n\varphi)\left(\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r\left(\frac{x_o}{x_d},\varphi,z\right)\right]_p dx_o\right)d\varphi \tag{5.3-18}$$

and

$$v_n(z) = -\frac{1}{2}\int_{-\pi}^{\pi}(\sin n\varphi)\left(\int_{x_h}^{1} G_s'(x_o)\left[\bar{i}_r\left(\frac{x_o}{x_d},\varphi,z\right)\right]_p dx_o\right) \tag{5.3-19}$$

and substituting for $(\bar{i}_r)_p$ from equation (5.2-8),

$$u_n(z) = -\frac{1}{2}\int_{x_h}^{1} G_s'(x_0)\int_{-\pi}^{\pi}(\cos n\varphi)\left[\bar{i}_r^{(1)} - \bar{i}_r^{(3)}\right]d\varphi\, dx_0 \qquad (5.3\text{-}20)$$

and

$$v_n(z) = -\frac{1}{2}\int_{x_h}^{1} G_s'(x_0)\int_{-\pi}^{\pi}(\sin n\varphi)\left[\bar{i}_r^{(1)} - \bar{i}_r^{(3)}\right]d\varphi\, dx_0 \qquad (5\text{-}3\text{-}21)$$

The integral for $\bar{i}_r^{(3)}$ in equation (5.3-20) will be discussed first. Substituting in from equation (5.2-6) this can be written

$$\int_{-\pi}^{\pi}(\cos n\varphi)\,\bar{i}_r^{(3)}\,d\varphi$$

$$= \int_{-\pi}^{\pi}\cos n\varphi\left[\frac{1}{x_d}\sum_{p=1}^{b}\left(\frac{2hz\sin(\varphi-\varphi_p)}{[4h^2z^2+\sin^2(\varphi-\varphi_p)]}\left[\frac{\left(\frac{x_0}{x_d}\right)-\cos(\varphi-\varphi_p)}{\sqrt{4h^2z^2+1+\left(\frac{x_0}{x_d}\right)^2-2\left(\frac{x_0}{x_d}\right)\cos(\varphi-\varphi_p)}} - \frac{\cos(\varphi-\varphi_p)}{\sqrt{4h^2z^2+1}}\right]\right)\right]d\varphi$$

$$= -\frac{2}{x_d}\left[\sum_{p=1}^{b}\sin\varphi_p\right]\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin 2n\theta\left(\frac{2hz\sin 2\theta}{[4h^2z^2+\sin^2 2\theta]}\left[\frac{\left(\frac{x_0}{x_d}\right)-\cos 2\theta}{\sqrt{4h^2z^2+\left(\frac{x_0}{x_d}+1\right)^2-4\left(\frac{x_0}{x_d}\right)\cos^2\theta}} - \frac{\cos 2\theta}{\sqrt{4h^2z^2+1}}\right]\right)d\theta$$

$$= -\frac{4hz}{x_d}\left[\sum_{p=1}^{b}\sin n\varphi_p\right]\left[\bar{i}_r^{(3)}\left(\frac{x_0}{x_d},z\right)\right]_n = 0 \qquad (5.3\text{-}22)$$

The equation has the value zero since

$$\sum_{p=1}^{b}\sin n\,\varphi_p = 0 \qquad (5\text{-}3\text{-}23)$$

for any number of blades.

If the Fourier coefficients $v_n(z)$ are being considered then

$$\int_{-\pi}^{\pi}\sin n\varphi\,\bar{i}_r^{(3)}\,d\varphi = -\frac{4hz}{x_d}\left[\sum_{p=1}^{b}\cos n\varphi_p\right]\left[\bar{i}_r^{(3)}\left(\frac{x_0}{x_d},z\right)\right]_n \qquad (5.3\text{-}24)$$

where

$$\sum_{p=1}^{b} \cos n\varphi_p = \begin{cases} b & \text{if} \quad n = mb \\ & \qquad m = 1, 2, 3 \cdots \\ 0 & \text{if} \quad n = mb \end{cases} \tag{5.3-25}$$

and

$$\begin{aligned} \left[i_r^{(3)}\left(\frac{x_o}{x_d}, Z\right) \right]_n &= (-1)^{n-1} \left(\frac{x_o}{x_d}\right)^{\frac{1}{2}} k \int_0^{\pi/2} \frac{\sin 2n\theta \sin 2\theta}{[4h^2 Z^2 + \sin^2 2\theta]} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ &+ (-1)^{n-1} \left(\frac{x_o}{x_d}\right)^{-\frac{1}{2}} k \int_0^{\pi/2} \frac{\sin 2n\theta \sin 2\theta \cos 2\theta \, d\theta}{[4h^2 Z^2 + \sin^2 2\theta] \sqrt{1 - k^2 \sin^2 \theta}} \\ &+ \frac{1}{\sqrt{4h^2 Z^2 + 1}} \int_0^{\pi/2} \frac{\sin 2n\theta \sin 2\theta \cos 2\theta \, d\theta}{[4h^2 Z^2 + \sin^2 2\theta]} \end{aligned} \tag{5.3-26}$$

$$k^2 = \frac{4\left(\frac{x_o}{x_d}\right)}{4h^2 Z^2 + \left(1 + \frac{x_o}{x_d}\right)^2} ; \quad k^2 \neq 1, \quad \text{For } k^2 = 1 \text{ equation (5.3-24) is zero}$$

An interesting observation from equations (5.3-22) and (5.3-24) is that the induced velocity from the bound vortex is an odd function and that many of the odd terms are zero. All "h" terms which are not multiples of the number of blades are zero.

The contribution of the free vortex system of the propeller to the Fourier coefficient $u_n(Z)$ is obtained from equations (5.3-20) and (5.2-3) as

$$\int_{-\pi}^{\pi} (\cos n\varphi) \, \bar{i}_r^{(1)} \, d\varphi$$

$$= \int_{-\pi}^{\pi} (\cos n\varphi) \left(\frac{x_o}{x_d}\right) \left(\frac{1}{x_d}\right) \left[\sum_{p=1}^{b} \int_{-\infty}^{0} \left(\frac{\left[2hZ - \alpha\left(\frac{x_o}{x_d}\right) \tan\beta_i\right] \cos(\varphi - \varphi_p - \alpha) - \alpha\left(\frac{x_o}{x_d}\right) \tan\beta_i \sin(\varphi - \varphi_p - \alpha)}{\left[1 + \left(\frac{x_o}{x_d}\right)^2 + \left(2hZ - \alpha \frac{x_o}{x_d} \tan\beta_i\right)^2 - 2\left(\frac{x_o}{x_d}\right) \cos(\varphi - \varphi_p - \alpha)\right]^{3/2}} \right) d\alpha \right] d\varphi$$

$$= \left(\frac{x_o}{x_d}\right) \left(\frac{1}{x_d}\right) \left[\sum_{p=1}^{b} \cos n\varphi_p \right] \left[\int_{-\pi}^{\pi} \cos n\bar{\theta} \left(\int_0^{\infty} \frac{\left[2hZ + \alpha\left(\frac{x_o}{x_d}\right) \tan\beta_i\right] \cos(\bar{\theta} + \alpha) - \left(\frac{x_o}{x_d}\right) \sin(\bar{\theta} + \alpha) \tan\beta_i}{\left[1 + \left(\frac{x_o}{x_d}\right)^2 + \left(2hZ + \alpha \frac{x_o}{x_d} \tan\beta_i\right)^2 - 2\left(\frac{x_o}{x_d}\right) \cos(\bar{\theta} + \alpha)\right]^{3/2}} d\alpha \right) d\bar{\theta} \right]$$

$$= \left(\frac{x_o}{x_d}\right)\left(\frac{1}{x_d}\right)\left[\sum_{p=1}^{b} \cos n\varphi_p\right]\left[\bar{f}_{nc}^{(1)}\left(\frac{x_o}{x},z\right)\right] \tag{5.3-27}$$

where

$$\left[\bar{f}_{nc}^{(1)}\left(\frac{x_o}{x_d},z\right)\right]$$
$$= \int_{-\pi}^{\pi} \cos n\bar{\theta}\left[\int_0^{\infty}\left(\frac{[2hz+\alpha(\;)\tan\beta_i]\cos(\bar{\theta}+\alpha)-\left(\frac{x_o}{x_d}\right)\tan\beta_i\sin(\theta+\alpha)}{\left[1+\left(\frac{x_o}{x_d}\right)^2+\left(2hz+\alpha\frac{x_o}{x_d}\tan\beta_i\right)^2-2\left(\frac{x_o}{x_d}\right)\cos(\bar{\theta}+\alpha)\right]^{3/2}}\right)d\alpha\right]d\bar{\theta} \tag{5.3-28}$$

By a similar analysis it can be shown that the contribution of the free vortex to the coefficient $v_n(z)$ is

$$\int_{-\pi}^{\pi}(\sin n\varphi)\,\bar{i}_r^{(1)}d\varphi = \left(\frac{x_o}{x_d}\right)\left(\frac{1}{x_d}\right)\left[\sum_{p=1}^{b}\cos n\varphi_p\right]\left[\bar{f}_{ns}\left(\frac{x_o}{x_d},z\right)\right] \tag{5.3-29}$$

where

$$\left[\bar{f}_{ns}\left(\frac{x_o}{x_d},z\right)\right] \tag{5.3-30}$$
$$= \int_{\pi}^{\pi}\sin n\bar{\theta}\left[\int_0^{\infty}\left(\frac{\left[2hz+\alpha\tan\beta_i\left(\frac{x_o}{x_d}\right)\right]\cos(\bar{\theta}+\alpha)-\left(\frac{x_o}{x_d}\right)\tan\beta_i\sin(\theta+\alpha)}{\left[1+\left(\frac{x_o}{x_d}\right)^2+\left(2hz+\alpha\frac{x_o}{x_d}\tan\beta_i\right)^2-2\left(\frac{x_o}{x_d}\right)\cos(\bar{\theta}+\alpha)\right]^{3/2}}\right)d\alpha\right]d\bar{\theta}$$

From equations (5.3-20), (5.3-22) and (5.3-27) and equations (5.3-21), (5.3-24) and (5.3-29), the Fourier coefficients for $n \neq 0$ are obtained as follows

$$u_n(z)' = \begin{cases} -\dfrac{b}{2x_d}\displaystyle\int_{x_h}^{1} G_s'(x_o)\left(\frac{x_o}{x_d}\right)\left[\bar{f}_{nc}^{(1)}\left(\frac{x_o}{x_d},z\right)\right] & \text{if } n = mb \\ & \quad m = 1,2,3\ldots \\ 0 \text{ if } n \neq mb & \end{cases} \tag{5.3-31}$$

and

$$v_n(z) = \begin{cases} \left(-\frac{b}{2x_d}\int_{x_h}^{1} G_i'(x_0)\frac{x_0}{x_d}\left[\bar{j}_{ns}^{(1)}\left(\frac{x_0}{x_d},z\right)\right] + 4hz\left[\bar{i}_{rn}^{(3)}\left(\frac{x_0}{x_d},z\right)\right]\right)dx_0 \ , & \text{if } n = mb \\ 0, & \text{if } n \neq mb \end{cases}$$

(5.2-32)

It will be noted that the free vortex system of the propeller contributes to both the odd and even terms while the lifting line vortex contributes to only the odd term.

The coefficients $\bar{i}_{rn}^{(3)}$ can easily be obtained but the coefficients $\bar{j}_{nc}^{(1)}$ and $\bar{j}_{ns}^{(1)}$ must be discussed farther. These last two coefficients are a special form of the equations discussed in Appendix E. The coefficient $\bar{j}_{nc}^{(1)}$, equation (5.3-28), is given by equation (E-18) where $\zeta = \dfrac{2hz}{x\ \tan\beta_i} = \dfrac{z_p}{x_o \tan\beta_i}$ and $\bar{j}_{ns}$ is given by equation (E-9) with the same value of ζ. It should be noted that $\bar{j}_{nc}^{(1)}$ has a logarithmic singularity at $x_o = x_d, z = 0$. This singularity can be removed by the method used for equation (5.3-14).

With these Fourier coefficients $u_n(z)$ and $v_n(z)$ the integral equation for the strength of the ring vortex distribution representing the duct, $\gamma(\varphi, z)$, can be solved. If the Fourier expansion of the ring vortex strength (5.3-3) is substituted into equation (5.3-1), an integral equation is obtained for the

Fourier coefficients, $g_n(z)$ and $h_n(z)$, of the vortex ring strength.

$$U(\varphi,Z) = h\int_0^1\oint_{-\pi}^{\pi}\left(\frac{\left[2h(\bar{z}-z')\cos(\varphi-\varphi')\right]\left[\sum_{n=1}^{\infty} g_n(z')\cos n\varphi' + \sum_{n=1}^{\infty} h_n(z')\sin n\varphi'\right]}{\left[4h^2(\bar{z}-z')^2 + 2\{1-\cos(\varphi-\varphi')\}\right]^{3/2}}\right)d\varphi' dz'$$

(5.3-33)

$$+ h\int_0^1\oint_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty} n h_n(z')\cos n\varphi' - \sum_{n=1}^{\infty} n g_n(z')\sin n\varphi'\right]\left(\int_c^{\infty}\frac{[2h(z-z')+\alpha\tan\beta_d]\cos(\varphi-\varphi'+\alpha) - \tan\beta_d\sin(\varphi-\varphi'+\alpha)}{\{[2h(z-z')+\alpha\tan\beta_d]^2 + 2[1-\cos(\varphi-\varphi'+\alpha)]\}^{3/2}}d\alpha\right)d\varphi' dz'$$

where

$\bar{z} = z - a_t$

The integral of each series in the above equation will now be considered separately. First, using the trigonometric identity; $1-\cos(\varphi-\varphi') = 2\sin^2\frac{1}{2}(\varphi-\varphi')$

$$h\int_0^1\oint_{-\pi}^{\pi}\left(\frac{\left[2h(\bar{z}-z')\cos(\varphi-\varphi')\right]\sum_{n=0}^{\infty} g_n(z')\cos n\varphi'}{\{4h^2(\bar{z}-z')^2 + 2[1-\cos(\varphi-\varphi')]\}^{3/2}}\right)d\varphi' dz' = h\sum_{n=0}^{\infty}\left(\int_0^1 g_n(z')[2h(\bar{z}-z')]\left[\int_{-\pi}^{\pi}\frac{\cos n\varphi'\cos(\varphi-\varphi')\,d\varphi'}{[4h^2(\bar{z}-z')^2 + 4\sin^2\frac{1}{2}(\varphi-\varphi')]^{3/2}}\right]dz'\right)$$

Now make the change of variable $\dfrac{\varphi-\varphi'}{2} = \theta$ and use the trigonometric identity

$$\cos n\varphi' = \cos n\varphi\cos 2n\theta + \sin n\varphi\sin 2n\theta$$

The last term integrates out over the integration range of $-\frac{\pi}{2} \leq \varphi' \leq \frac{\pi}{2}$

$$2h\sum_{n=0}^{\infty}\left(\int_0^1 g_n(z')[2h(\bar{z}-z')]\left[\int_{-\frac{\pi}{2}-\frac{\varphi}{2}}^{\frac{\pi}{2}-\frac{\varphi}{2}}\frac{(\cos n\varphi\cos 2n\bar{\theta} + \sin n\varphi\sin 2n\bar{\theta})\cos 2\bar{\theta}}{[4h^2(\bar{z}-z')^2 + 4\sin^2\bar{\theta}]^{3/2}}d\bar{\theta}\right]dz'\right)$$

$$= 4h \sum_{n=1}^{\infty} \cos n\varphi \int_0^1 g_n(z')[2h(\bar{z} - z')] \left[\int_0^{\pi/2} \frac{\cos 2n\bar{\theta} \cos 2\bar{\theta} \, d\theta}{[4h^2(\bar{z} - z')^2 + 4\sin^2\bar{\theta}]^{3/2}} \right] dz'$$

The integrand of the integral is periodic with respect to θ so it makes no difference whether the integration is carried out over the range $-\frac{\pi}{2} - \frac{\varphi}{2} \leq \bar{\theta} \leq \frac{\pi}{2} - \frac{\varphi}{2}$ or $-\frac{\pi}{2} \leq \bar{\theta} \leq \frac{\pi}{2}$ and since the integrand is an even function, the integral over the range from 0 to $\frac{\pi}{2}$ is twice that from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The integral in the brackets can be reduced to a function tabulated by Riegels[37] by making a change in variable $\bar{\theta} = \theta + \frac{\pi}{2}$ and using the trigonometric identity

$$2\cos 2n\bar{\theta} \cos 2\bar{\theta} = \cos 2(n+1)\bar{\theta} + \cos 2(n-1)\bar{\theta}$$

$$2h \sum_{n=1}^{\infty} \cos n\varphi \int_0^1 g_n(z')[2h(\bar{z} - z')] \left[\int_0^{\pi/2} \frac{[\cos 2(n+1)\bar{\theta} + \cos 2(n-1)\bar{\theta}] \, d\bar{\theta}}{[4h^2(\bar{z} - z')^2 + 4\sin^2\bar{\theta}]^{3/2}} \right] dz'$$

$$= h \sum_{n=1}^{\infty} \cos n\varphi \int_0^1 g_n(z')[2h(\bar{z} - z')] \frac{k^3}{4} \left(\int_0^{\pi/2} \frac{[(-1)^{n+1}\cos 2(n+1)\theta + (-1)^{n-1}\cos 2(n-1)\theta] \, d\theta}{[1 - k^2 \sin^2\theta]^{3/2}} \right) dz'$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \cos n\varphi \int_0^1 \frac{g_n(z')}{(\bar{z} - z')} k^3 \left\{ [h(\bar{z} - z')]^2 [G_{n+1}(k) + G_{n-1}(k)] \right\} dz' \qquad (5.3\text{-}34)$$

where

$$k^2 = \frac{1}{h^2(\bar{z} - z')^2 + 1}$$

and

$$G_n(k) = (-1)^n \int_0^{\pi/2} \frac{\cos 2n\theta}{[1 - k^2 \sin^2\theta]^{3/2}} \, d\theta \qquad (5.3\text{-}35)$$

Riegels[37] has given the expansion of this function $G_n(k)$ as $k \to 1$, which is when $z' \to \bar{z}$, as

$$G_n(k) = \left(\frac{1}{1-k^2}\right) - b_n \ln \sqrt{1-k^3} + \ldots \quad (5.3\text{-}36)$$

thus $G_n(k)$ has a singularity as $k \to 1$. This singularity is removed by multiplying by $(\bar{z} - z')^2$.

$$\lim_{z' \to \bar{z}} \left[h^2 (\bar{z} - z')^2 G_n(k) \right] = \lim_{z' \to \bar{z}} \left[\frac{h^2}{1} \frac{(\bar{z} - z')^2}{k^2} - b_n h^2 (\bar{z} - z')^2 \ln \sqrt{1 - k^2} \right]$$

$$= \lim_{z' \to \bar{z}} \left(\frac{h^2 (\bar{z} - z')^2}{h^2 (\bar{z} - z')^2} \left[h^{2} (\bar{z} - z') + 1 \right] \right) = 1$$

From this limit it can be seen that the function within the braces of equation (5.3-34) is not singular as $z' \to \bar{z}$, but the complete integrand has a singularity at $z' = \bar{z}$. Values of $G_0(k)$ and $G_1(k)$ can easily be obtained in the form of complete elliptic integrals. These two values are

$$G_0(k) = \frac{E(k)}{1 - k^2}$$

$$G_1(k) = \frac{(2 - k^2) K(k) - 2(1 - k^2) K(k)}{k^2 (1 - k^2)} \quad (5.3\text{-}37)$$

The asymptotic expansion of $G_n(k)$ for $k \ll 1$ is given by Riegels[37] as

$$G_n(k) = \frac{\pi}{2} \frac{(2n+1)!}{2^{2n} n! n!} k^{2n} \quad (5.3\text{-}38)$$

The integral involving the series $\sum_{n=1}^{\infty} h_n(z') \sin n\varphi$ in

equation (5.3-33) reduces in a similar manner to the following equation

$$h\int_0^1\int_{-\pi}^{\pi}\left(\frac{[2h(\bar z-z')\cos(\varphi-\varphi')]\sum_{n=1}^{\infty}h_n(z')\sin n\varphi'}{\{4h^2(\bar z-z')^2+2[1-\cos(\varphi-\varphi')]\}^{3/2}}\right)d\varphi' dz'$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\sin n\varphi\int_0^1\frac{h_n(z')}{(\bar z-z')}k^3\left|h^2(\bar z-z')[G_{n+1}(k)+G_{n-1}(k)]\right|dz' \tag{5.3-39}$$

The other terms occurrin· in equation (5.3-33) are somewhat more complicated since the helical vortex sheet shed from the duct gives rise to both odd and even terms. This is shown in the following reduction of the integral involving $g_n(z')$. In this reduction the order of integration is interchanged, the change of variable $\theta=\varphi-\varphi'$ made, and the order of integration changed back again.

$$\sum_{n=1}^{\infty}nh\int_0^1 g_n(z')\int_{-\pi}^{\pi}\sin n\varphi'\left(\int_0^{\infty}\left[\frac{[2h(\bar z-z')+\alpha\tan\beta_d]\cos(\varphi-\varphi'+\alpha)-\tan\beta_d\sin(\varphi-\varphi'+\alpha)}{\{[2h(\bar z-z')+\alpha\tan\beta_d]^2+2[1-\cos(\varphi-\varphi'+\alpha)]\}^{3/2}}\right]d\alpha\right)d\varphi' dz'$$

$$=\sum_{n=1}^{\infty}nh\int_0^1 g_n(z')\int_{-\pi}^{\pi}\left(\sin n\varphi\cos n\theta-\cos n\varphi\sin n\theta\right)\int_0^{\infty}\left[\frac{[2h(\bar z-z')+\alpha\tan\beta_d]\cos(\theta+\alpha)-\tan\beta_d\sin(\theta+\alpha)}{\{[2h(\bar z-z')+\alpha\tan\beta_d]^2+4\sin^2\frac{1}{2}(\theta+\alpha)\}^{3/2}}\right]d\alpha\, d\theta\, dz'$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\sin n\varphi\left(\int_0^1\frac{g_n(z')}{\bar z-z'}i_n^{(c)}(\bar z-z')dz'\right)-\frac{1}{2}\sum_{n=1}^{\infty}\cos n\varphi\left(\int_0^1\frac{g_n(z')}{\bar z-z'}i_n^{(s)}(\bar z-z')dz'\right) \tag{5.3-40}$$

where

$$i_n^{(c)}(\bar{z}-z') = n[2h(\bar{z}-z')]\int_{-\pi}^{\pi}\cos n\theta\left(\int_0^{\infty}\left[\frac{[2h(\bar{z}-z')+\alpha\tan\beta_d]\cos(\theta+\alpha)-\tan\beta_d\sin(\theta+\alpha)}{\{[2h(\bar{z}-z')+\alpha\tan\beta_d]^2+4\sin^2\frac{1}{2}(\theta+\alpha)\}^{3/2}}\right]d\alpha\right)d\theta \tag{5.3-41}$$

and

$$i_n^{(s)}(\bar{z}-z') = n[2h(\bar{z}-z')]\int_{-\pi}^{\pi}\sin n\theta\left(\int_0^{\infty}\left[\frac{[2h(\bar{z}-z')+\alpha\tan\beta_d]\cos(\theta+\alpha)-\tan\beta_d\sin(\theta+\alpha)}{\{2h(\bar{z}-z')+\alpha\tan\beta_d]^2+4\sin^2\frac{1}{2}(\theta+\alpha)\}^{3/2}}\right]d\alpha\right)d\theta \tag{5.3-42}$$

The integrals in $i_n^{(c)}$ and $i_n^{(s)}$, equations (5.3-41) and (5.3-42) respectively, have been reduced to a simpler form in Appendix E by evaluating the infinite integral. Using Appendix E equations (5.3-41) and (5.3-42) become

$$i_n^{(c)}(\bar{z}-z') = n[2h(\bar{z}-z')]I_2' \tag{5.3-43}$$

$$i_n^{(s)}(\bar{z}-z') = n[2h(\bar{z}-z')]I_1' \tag{5.3-44}$$

where I_1' and I_2' are given by equations (E-9) and (E-18) respectively with $\bar{x}=1$.

The integral from the vortex sheet involving the Fourier coefficient $h_n(z')$ is reduced in similar manner and is

$$\sum_{n=1}^{\infty} nh\int_0^1 h_n(z')\int_{-\pi}^{\pi}\cos n\varphi'\left(\int_0^{\infty}\left[\qquad\right]d\alpha\right)d\varphi' dz' \tag{5.3-45}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\cos n\varphi\left(\int_0^1\frac{h_n(z')}{\bar{z}-z'}i_n^{(c)}(\bar{z}-z')dz'\right) = \frac{1}{2}\sum_{n=1}^{\infty}\sin n\varphi\left(\int_0^1\frac{h_n(z')}{\bar{z}-z'}i_n^{(s)}(\bar{z}-z')dz'\right)$$

If the Fourier series for $U(\varphi,\bar{z})$ is substituted into equations (5.3-33) along with equations (5.3-34), (5.3-39) (5.3-40) and (5.3-45), the equation is written completely in terms of a Fourier series. Since a Fourier series is unique and linear, the coefficients can be equated. The result is two linear singular integral equations for the Fourier coefficients of the ring vortex strength, $g_n(z)$ and $h_n(z)$, in terms of the known coefficients $u_n(z)$ and $v_n(z)$, equations (5.3-31) and (5.3-32).

$$\left.\begin{aligned} u_n(\bar{z}) &= \frac{1}{2}\int_0^1 \frac{1}{\bar{z}-z'}\left[g_n(z')M_n(\bar{z}-z') + h_n(z')i_n^{(c)}(\bar{z}-z')\right]dz' \\ v_n(\bar{z}) &= \frac{1}{2}\int_0^1 \frac{1}{\bar{z}-z'}\left[-g_n(z')i_n^{(c)}(\bar{z}-z') + h_n(z')M_n(\bar{z}-z')\right]dz' \end{aligned}\right\} \quad (5.3\text{-}46)$$

where

$$M_n(\bar{z}-z') = k^3\left|[h(\bar{z}-z')]^2[G_{n+1}(k) + G_{n-1}(-k)]\right| + i_n^{(s)}(\bar{z}-z') \quad (5.3\text{-}47)$$

From the equations for $u_n(\bar{z})$ and $v_n(\bar{z})$ it is known that $u_n(\bar{z}) = v_n(\bar{z})=0$ if $n\neq mb$, $(m=1,2....)$. Since equation (5.3-44) must hold for all values of z, it must be concluded that

$$g_n(\bar{z}) = h_n(\bar{z})=0 \quad \text{if } n\neq mb, \ (m=1,2.....)$$

because the coefficients $K_n(\bar{z}-z')$ and $i_n^{(c)}(\bar{z}-z')$ obviously are not zero for $n\neq mb$. From a practical point of view this fact greatly reduces the number coefficients which must be calculated. For instance for a 4-bladed propeller in a duct

only $g_4(\bar{z})$, $h_4(\bar{z})$, $g_8(\bar{z})$, etc. exists. (It should be noted that $g_0(\bar{z})$ exists for all number of blades and that $h_0(\bar{z})$ is identically zero). From the decrease in the order of magnitude of Fourier coefficients[28] with increasing n, it is not difficult to conclude that the series for the circulation distribution converges very rapidly since so many of the terms are identically zero.

V.4 Reduction of the Integral Equation for the Duct Ring Vortex Strength.

The method of solution of the integral equations (5.3-46) will now be considered. First, however, the solution when $n = 0$ will be discussed. For this case $h_o(z) = v_o(z) = i_o^{(e)}(\bar{z} - z') = 0$ and the system of equations reduces to a singular integral equation for $g_o(z')$.

$$u_o(\bar{z}) = \int_0^1 \frac{g_o}{\bar{z} - z'} K_o(\bar{z} - z')dz' \tag{5.4-1}$$

where

$$K_o(\bar{z} - z') = -g(\bar{z} - z') = -k\left(4h^2(\bar{z} - z')^2\left[K(k) - E(k)\right] - 2E(k)\right) \tag{2.4-2}$$

The function $u_o(\bar{z})$ is given by equation (5.3-16). This equation (5.4-1) is identical to the one solved in section II.4 with the additional terms from the induced radial velocities from the hub and propeller. The solution method is the same except that additional terms now occur in equations (2.6-2) and (2.6-6) for the function $f(\theta)$. Following the same procedure as in section II.4, the functions $H_h(\bar{z})$ and $H_p(\bar{z})$, which occur in the equation for $u_o(\bar{z})$, are expanded in a Fourier cosine series in θ, for $0 \leq \theta \leq \pi$, where $\bar{z} = z - a_t = (1/2)(1 + \cos\theta)$, then

$$\left[H_h(\theta) + H_p(\theta)\right] = \sum_{m=0}^{\infty} p_m \cos m\theta \tag{5.4-2}$$

where

$$p_0 = \frac{1}{\pi}\int_0^{\pi}[H_h(\theta') + H_p(\theta')]d\theta'$$

$$p_m = \frac{2}{\pi}\int_0^{\pi}[H_h(\theta') + H_p(\theta')]\cos m\theta' d\theta'$$

(5.4-3)

and $H_p(\theta)$ and $H_h(\theta)$ are obtained from equation (5.3-16) with the change of variable mentioned previously. Substituting equation (5.4-2) into equation (2.4-14), the new function $f_p(\theta)$ is obtained

$$f_p(\theta) = f(\theta) + \frac{1}{2\pi}\left[p_0 \cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\sum_{m=1}^{\infty} p_m \sin m\theta\right] \qquad (5.4\text{-}4)$$

where $f(\theta)$ is given by equation (2.4-26).

This function $f_p(\theta)$ is used in equations (2.6-2) and (2.6-6) in place of $f(\theta)$. This equation (5.4-4) is for the general case, i.e. when a singularity occurs in the circulation distribution at the leading edge. It also seems reasonable to describe an ideal angle of attack of the duct section when the propeller is in the duct. Obviously, each section of a symmetrical duct cannot operate at an ideal angle of attack in the presence of the propeller but an ideal angle of attack can be defined in presence of the average velocity. Making the definition of ideal angle of attack with the propeller in the duct as the angle of attack in which the singularity in $g_0(z)$ does not occur at the leading edge of the duct, the following is written for the function $f_{pid}(\theta)$.

$$f_{pid}(\theta) = f_{id}(\theta) - \frac{1}{2\pi}\sum_{m=1}^{\infty} p_m \sin m\theta \qquad (5.4\text{-}5)$$

where f_{id} is obtained from equation (2.5-3).

To obtain the ideal circulation distribution $[g_o(z)]_{id}$ in presence of the average hub and propeller induced velocities, the foregoing equation is used for $f(\theta)$ in equations (2.6-2) and (2.6-6) and the calculations carried out as described in section II.6 for the ideal case.

By the procedure just described the Fourier coefficient g_o or g_o^*, where $g_o^*(\theta) = g_o(\theta) \sin \frac{1}{2}\theta$, can be obtained by the methods already described in sections II.4, II.5 and II.6. The Fourier coefficients $g_n(\bar{z})$ and $h_n(\bar{z})$ for $n \neq 0$ cannot be obtained quite so easily because they are defined by two linear singular integral equations and this involves the solution of a system of singular integral equations.

The system of singular integral equations given by equation (5.3-46) can be reduced to a system of non-singular Fredholm equations of the second kind which can be evaluated by known methods[29]. The method of reduction is the same as given previously in that the singular integral equation is reduced to the airfoil equation and then inverted. Starting with the first equation of (5.3-46) the term $M_n(\bar{z}-\bar{z}) = M_n(0)$ is added and subtracted from the integrand. It can easily be shown from equations (5.3-43) and (5.3-47) that $M_n(0) = 2$, and $i_n^{(c)}(0) = 0$.

$$\int_0^1 \frac{g_n(z')}{(\bar{z}-z')} dz' = u_n(\bar{z}) + \frac{1}{2}\int_0^1 \frac{1}{\bar{z}-z'}\left[g_n(z')(2-M_n) - h_n(z')\, i_n^{(c)}(0)\right] dz' \qquad (5.4\text{-}6)$$

This equation is now inverted as in section II.4. The Kutta condition, equation(2.2-21), is satisfied by making $g_n(\bar{z})$ zero at the trailing edge of the duct. Inverting and interchanging the order of integration the previous equation can be written as a Fredholm equation of the second kind.

$$g_n(\bar{z}) = \frac{1}{\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\oint_0^1 \frac{1}{(z'-\bar{z})}\sqrt{\frac{1-z'}{z'}}\, u_n(z')dz'$$

$$+ \frac{1}{2\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\int_0^1\left(\oint_0^1\sqrt{\frac{1-z''}{z''}}\left[\frac{2-M_n(z''-z')}{(z''-\bar{z})(z''-z')}\right]dz''\right) g_n(z')dz'$$

$$- \frac{1}{2\pi^2}\sqrt{\frac{\bar{z}}{(1-\bar{z})}}\int_0^1\left(\int_0^1\sqrt{\frac{1-z''}{z''}}\left[\frac{i_n^{(c)}(z''-z')}{(z''-\bar{z})(z''-z')}\right]dz''\right) h_n(z')dz' \qquad (5.4\text{-}7)$$

A singularity occurs at $\bar{z} = 1$ which is the leading edge of the duct. As discussed previously, the singularity is removed by introducing a new variable $g_n^*(\bar{z}) = \sqrt{1-\bar{z}}\, g_n(\bar{z})$ also $h_n^*(\bar{z}) = \sqrt{1-\bar{z}}\, h_n(\bar{z})$. If the change of variable $\bar{z} = (1/2)(1+\cos\theta)$, $z' = (1/2)(1+\cos\theta')$, etc., is made, the preceding equation becomes

$$g_n^*(\theta) = g_n(\theta)\sin\tfrac{1}{2}\theta = f_n(\theta) + \int_0^\pi\left[K_n(\theta,\theta')g_n^*(\theta') - \bar{K}_n(\theta,\theta')h_n^*(\theta')\right]d\theta \qquad (5.4\text{-}8)$$

where

$$f_n(\theta) = \frac{1}{\pi^2}\cos\tfrac{1}{2}\theta\oint_0^\pi\frac{(1-\cos\theta')}{(\cos\theta'-\cos\theta)}\,u_n(\theta')d\theta' \qquad (5.4\text{-}9)$$

$$K_n(\theta,\theta') = \frac{1}{\pi^2}\cos\frac{1}{2}\theta'\cos\frac{1}{2}\theta \int_0^{\pi} \frac{(1-\cos\theta'')}{(\cos\theta''-\cos\theta)}\left[\frac{2-M_n(\cos\theta''-\cos\theta')}{(\cos\theta''-\cos\theta')}\right] d\theta'' \tag{5.4-10}$$

$$\bar{K}_n(\theta,\theta') = \frac{1}{\pi^2}\cos\frac{1}{2}\theta'\cos\frac{1}{2}\theta \int_0^{\pi} \frac{(1-\cos\theta'')}{(\cos\theta''-\cos\theta)}\left[-\frac{i_n^{(c)}(\cos\theta''-\cos\theta')}{(\cos\theta''-\cos\theta')}\right] d\theta'' \tag{5.4-11}$$

The last three integrals are evaluated by expanding the integrands in a Fourier cosine series in θ' or θ" and using the Cauchy principal value integrals given by equation (2.4-25), then

$$f_n(\theta) = \frac{1}{\pi}\left[-a_{no}\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\sum_{m=1}^{\infty} a_{nm}\sin m\theta\right. \tag{5.4-12}$$

where

$$a_{no} = \frac{1}{\pi}\int_0^{\pi} u_n(\theta')d\theta' \tag{5.4-13}$$

$$a_{nm} = \frac{2}{\pi}\int_0^{\pi} u_n(\theta')\sin m\theta' d\theta' \tag{5.4-14}$$

$$K(\theta,\theta') = \frac{1}{\pi}\cos\frac{1}{2}\theta'\left[-b_{no}(\theta')\cos\frac{1}{2}\theta + (\sin\frac{1}{2}\theta)\sum_{m=1}^{\infty} b_{nm}(\theta')\sin m\theta\right] \tag{5.4-15}$$

where

$$b_{no}(\theta') = \frac{1}{\pi}\int_0^{\pi}\left[\frac{2-M_n}{\cos\theta''-\cos\theta'}\right]d\theta'' \tag{5.4-16}$$

$$b_{nm}(\theta') = \frac{2}{\pi}\int_{0}^{\pi}\left[\frac{2 - M_n}{\cos\theta'' - \cos\theta'}\right] \cos m\theta'' d\theta' \tag{5.4-17}$$

and

$$\bar{K}_n(\theta,\theta') = \frac{1}{\pi} \cos\frac{1}{2}\theta'\left[-d_{no}(\theta')\cos\frac{1}{2}\theta + (\sin\frac{1}{2}\theta)\sum_{m=1}^{\infty} d_{nm}(\theta')\sin m\theta\right] \tag{5.4-18}$$

where

$$d_{no}(\theta') = \frac{2hn}{\pi}\int_{0}^{\pi} I_2'(\cos\theta'' - \cos\theta')d\theta'' \tag{5.4-19}$$

$$d_{nm}(\theta') = \frac{4hn}{\pi}\int_{0}^{\pi} I_2'(\cos\theta'' - \cos\theta')\cos m\theta' d\theta' \tag{5.4-20}$$

The function I_2' is given by equation (E-18) with $\bar{x} = 1$. It has a logarithmic singularity at $\theta'' = \theta'$ which can easily be removed. The integrand of the integrals for the Fourier coefficients b_{no} and b_{nm}, equation (5.4-16) and (5.4-17), are of indeterminate form. Referring to equation (5.3-47), it can easily be shown that

$$\lim_{\theta''\to\theta'}\left[\frac{2 - M_n}{\cos\theta'' - \cos\theta'}\right] = 2\, n\, h\, I_1'$$

The integral I_1' is given by equation (E-9), Appendix E, with $\bar{x} = 1$ and the proper change of variables.

The second integral equation of the system, equation (5.3-20), can be reduced to a Fredholm equation of the

second kind for the coefficient $h_n^*(\theta)$ in the same manner. Repeating equation (5.4-7) the coupled Fredholm equations of the second kind are

$$g_n^*(\theta) = f_n(\theta) + \int_0^{\pi} K_n(\theta,\theta') g_n^*(\theta') d\theta' - \int_0^{\pi} \bar{K}_n(\theta,\theta') h_n^*(\theta') d\theta'$$

$$h_n^*(\theta) = \hat{f}_n(\theta) + \int_0^{\pi} \bar{K}_n(\theta,\theta') g_n^*(\theta') d\theta' + \int_0^{\pi} K_n(\theta,\theta') h_n^*(\theta') d\theta' \tag{5.4-21}$$

where

$$\hat{f}_n(\theta) = \frac{1}{\pi}\left[-\hat{a}_{no}\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta \sum_{m=1}^{\infty} \hat{a}_{nm} \sin m\theta\right. \tag{5.4-22}$$

$$\hat{a}_{no} = \frac{1}{\pi}\int_0^{\pi} v_n(\theta') d\theta' \tag{5.4-23}$$

$$\hat{a}_{nm} = \frac{2}{\pi}\int_0^{\pi} v_n(\theta') \sin m\theta' d\theta' \tag{5.4-24}$$

also

$$g_n^*(\theta) = g_n(\theta)\sin\tfrac{1}{2}\theta \quad \text{and} \quad h_n^*(\theta) = h_n(\theta)\sin\tfrac{1}{2}(\theta)$$

The system of equations (5.4-21) in the interval $(0,\pi)$ can be reduced to a single equation in the interval $(0,2\pi)$ as discussed in Reference [29]. This single equation is not necessarily more convenient to use than applying the various solution methods directly to the system of equations (5.4-21).

To obtain a numerical solution to the system of equations, they are rewritten in the following form

$$g_n^*(\theta) = F_1(\theta) + \frac{\sin\frac{1}{2}\theta}{\pi}\left[-A_0\cos\tfrac{1}{2}\theta + A_1\sin\theta + \cdots + A_m\sin m\theta\right] \tag{5.4-25}$$

$$h_n^*(\theta) = F_2(\theta) + \frac{\sin\frac{1}{2}\theta}{\pi}\left[-B_0\cot\tfrac{1}{2}\theta + B_1\sin\theta + \cdots\ B_m\sin m\theta\right] \tag{5.4-26}$$

where

$$F_1(\theta) = f_n(\theta) - \int_0^{\pi}\bar{K}_n(\theta,\theta')h_n^*(\theta')d\theta' \tag{5.4-27}$$

$$F_2(\theta) = \hat{f}_n(\theta) + \int_0^{\pi}\bar{K}(\theta,\theta')g_n^*(\theta')d\theta' \tag{5.4-28}$$

$$A_m = \int_0^{\pi} b_{nm}(\theta')\cos\tfrac{1}{2}\theta'\, g_n^*(\theta')d\theta' \tag{5.4-29}$$

$$B_m = \int_0^{\pi} b_{nm}(\theta')\cos\tfrac{1}{2}\theta'\, h_n^*(\theta')d\theta' \tag{5.4-30}$$

The Fourier coefficient b_{nm} is given by equation (5.4-16) or (5.4-17) and since they are functions of geometry only, they can be tabulated. Each of the equations (5.4-25) and (5.4-26) can be solved by the method given in section II.6. The problem now exists that F_1 and F_2 are unknown, consequently, it is necessary to use the method of successive substitution.

Equation 5.4-25 is first solved assuming $F_1(\theta) = f_n(\theta)$ and then the resulting $g_n^*(\theta)$ used to solve equation (5.4-26). This process is repeated using each time new values of $F_1(\theta)$ and $F_2(\theta)$ until satisfactory convergence is obtained. All the Fredholm theorems apply to the system of equations so, in general, convergence is assured.

Once the Fourier coefficients g_n^* and h_n^* have been determined, the circulation distribution of the ring vortex strength γ can be calculated from equation (5.3-3) and the induced velocities at the propeller and hub can be determined. In general, it will be found advantageous to make use of the Fourier series expansion of the ring vortex strength.

V.5 Induced Drag of the Duct

In Section II.7 the induced drag of the duct was expressed in the form of a duct thrust coefficient.

$$(C_{T_{si}})_d = \frac{4h\chi_d^2}{\pi}\int_0^1\int_0^{2\pi}\gamma(\varphi,\bar{z})\left[\frac{w_r}{V_s}(x_d,\varphi,\bar{z})_{p+h}+\frac{w_r}{V_s}(x_d,\varphi,\bar{z})_{\frac{\partial\gamma}{\partial\varphi}}\right]d\varphi\, d\bar{z} \tag{2.7-8}$$

$$= 4h\chi_d^2\left[C_{T_1}+C_{T_2}+C_{T_3}\right] \tag{5.5-1}$$

where

$$C_{T_1} = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\gamma(\varphi,\bar{z})\frac{w_r}{V_s}(x_d,\bar{z})_h\, d\varphi\, d\bar{z} \tag{5.5-2}$$

$$C_{T_2} = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\gamma(\varphi,\bar{z})\frac{w_r}{V_s}(x_d,\varphi,\bar{z})_p\, d\varphi\, d\bar{z} \tag{5.5-3}$$

$$C_{T_3} = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\gamma(\varphi,\bar{z})\frac{w_r}{V_s}(x_d,\varphi,\bar{z})_{\frac{\partial\gamma}{\partial\varphi}}\, d\varphi\, d\bar{z} \tag{5.5-4}$$

The radial velocity induced by the hub at the duct is given by equation (5.2-9). If this equation is substituted into equation (5.5-2), the contribution to the duct thrust by the hub is given. This equation can be considerably simplified if the Fourier expansion of γ is used, then

$$C_{T_1} = 2\int_0^1 g_0(\bar{z})\frac{w_r}{V_s}(x_d,\bar{z})_h\, d\bar{z}$$

Since $g_0(\bar{z})$ may be singular at the leading edge, the change of variable $\bar{z}=\frac{1}{2}(1+\cos\theta)$ is introduced and then the

equation can be written in terms of the coefficient $g_o^*(\theta)$ which was discussed in the last section.

$$C_{T_1} = 2\int_0^{\pi} g_o^*(\theta)\frac{w_r}{V_s}(x_d,\theta)_h \cos\tfrac{1}{2}\theta d\theta \tag{5.5-5}$$

where

$$g_o^*(\theta)=\sin\tfrac{1}{2}\theta\, g_o(\theta)$$

The radial velocity induced on the duct by the propeller is given by equation (5.2-8). If the Fourier series for both γ and the radial velocity induced by the propeller is used, equation (5.5-3) is simplified to

$$C_{T_2}= \frac{1}{2\pi}\int_0^{\pi} g^*(\theta)\bar{u}_o(\theta)\cos\tfrac{1}{2}\theta d\theta - \frac{1}{2\pi}\int_0^{\pi}\left[\sum_{n=1}^{\infty} g_n^*(\theta)u_n(\theta)\right]\cos\tfrac{1}{2}\theta d\theta$$
$$- \frac{1}{2\pi}\int_0^{\pi}\left[\sum_{n=1}^{\infty} h_n^*(\theta)v_n(\theta)\right]\cos\tfrac{1}{2}\theta d\theta \tag{5.5-6}$$

The function $\bar{u}_o(\theta)$ is given by equation (5.3-15) with the change in variable, $z=\frac{1}{2}(1+\cos\theta)+a_t$. The functions $g_n^*(\theta)$, $h_n^*(\theta)$, $u_n(\theta)$ and $v_n(\theta)$ are the same as discussed in the previous section with the necessary change in variable.

The radial velocity induced on the duct by the duct trailing vortex system is given by equation (5.1-5). After introducing the Fourier expansions for γ and making use of the evaluation of the infinite integral in Appendix E, the contribution to the nozzle thrust by the trailing vortex system

is

$$C_{T_3} = \frac{h}{2\pi}\int_0^{\pi}\left(\sum_{n=1}^{\infty} n g_n^*(\theta)\left[\int_0^{\pi} g_n^*(\theta') I_1' \cos\tfrac{1}{2}\theta' d\theta' + \int_0^{\pi} h_n^*(\theta) I_2' \cos\tfrac{1}{2}\theta' d\theta'\right]\cos\tfrac{1}{2}\theta\right) d\theta$$

$$- \frac{h}{2\pi}\int_0^{\pi}\left(\sum_{n=1}^{\infty} n h_n^*(\theta)\left[\int_0^{\pi} g_n^*(\theta') I_2' \cos\tfrac{1}{2}\theta' d\theta' - \int_0^{\pi} h_n^*(\theta') I_2' \cos\tfrac{1}{2}\theta' d\theta'\right]\cos\tfrac{1}{2}\theta\right) d\theta \tag{5.5-7}$$

The functions I_1' and I_2' are given by equations (E-9) and (E-11) with the variable $\zeta = \frac{2h}{\tan\beta_d}(\cos\theta - \cos\theta')$, $\beta_o = \beta_d$ and $\bar{x} = 1$.

Once the contribution to the duct thrust by the various components have been obtained the total duct thrust follows easily. It should be remembered that only terms which are harmonic with the number of blades appear in the series in these equations, consequently a majority of the terms are zero.

V.6. Velocities Induced by the Duct and Its Trailing Vortex System at the Propeller and Hub.

Only the axial and tangential velocities induced at a propeller lifting line are desired. Since the flow is symmetrical, it is sufficient to consider only the lifting line at $\varphi = 0$. The velocity induced at the propeller by the duct can be considered as that due to the ring vortices, ring sources and duct trailing vortex system.

$$\left[\frac{w_a}{V_s}(x,0,0)\right]_d = \frac{w_a}{V_s}(x,0)_q + \frac{w_a}{V_s}(x,0,0)_\gamma + \frac{w_a}{V_s}(x,0,0)_{\frac{\partial\gamma}{\partial\varphi}} \tag{5.6-1}$$

and

$$\left[\frac{w_t}{V_s}(x,0,0)\right]_d = \frac{w_t}{V_s}(x,0,0)_\gamma + \frac{w_t}{V_s}(x,0,0)_{\frac{\partial\gamma}{\partial\varphi}} \tag{5.6-2}$$

The ring source system does not induce a tangential velocity since its strength is independent of angle. The axial velocity induced by the source ring is given by equation (B-12). Introducing the source strength in terms of the thickness slope of the duct, equation (2.3-4), the axial velocity induced by the source ring is obtained as

$$\frac{w_a}{V_s}(x,0)_q = \frac{h}{2\pi}\left(\frac{x_d}{x}\right)^{3/2}(1 - w_x)\int_0^1 S'(z')\left[2h(a_t + z')\right]\left[\frac{k_2^3 E(k_2)}{1 - k_2^2}\right]dz' \tag{5.6-3}$$

where

$$k_2^2 = \frac{4\left(\frac{x}{x_d}\right)}{4h^2(a_t + z')^2 + \left(\frac{x}{x_d} + 1\right)^2} \tag{5.6-4}$$

The integrand of this integral is singular at the point $x = x_d$ and $z' = -a_t$. The axial position $z' = -a_t$ is at the lifting line and $x = x_d \doteq 1$ only occurs at the blade tip. When the propeller is not within the duct then the singularity does not exist, i.e. $z' = -a_t$ does not lie between 0 and 1. When the singularity exist this equation must be treated as a Cauchy principal value integral. Considering the case t[illegible] $x = x_d$ then the precedi[illegible] equation can be expressed as

$$\frac{w_a}{V_s}(1,0)_g = \frac{1}{\pi}(1 - w_{xd}) \oint_0^1 S'(z')\bar{k}_2 \frac{E(\bar{k}_2)}{(a_t+z')}\, dz' \tag{5.6-5}$$

where

$$\bar{k}_2^2 = \frac{\text{[illegible]}}{h^2(a_t+z)^2+1}$$

If the change of variable is made, $z' = (1/2)(1 + \cos\theta)$ then

$$\frac{w_a}{V_s}(1,0)_g = \frac{1}{\pi}(1 - w_{xd}) \oint_0^\pi S'(\theta)\bar{k}_2 \frac{E(\bar{k}_2)}{(\cos\theta - \cos\theta_t)} \sin\theta\, d\theta \tag{5.6-6}$$

where

$$\cos\theta_t = -(2a_t + 1)$$

$$\bar{k}_2^2 = \frac{4}{h^2(\cos\theta' - \cos\theta_t)^2 + 4}$$

The term, $s'(\theta)\bar{k}_2\, E(\bar{k}_2) \sin\theta$, is expanded in a Fourier cosine series in θ over the range 0 to π. This involves no additional assumptions on $s'(\theta)$ than in Section II.4. If this is done, and the series substituted into the above equation and the integration carried out, the induced velocity is

finally given as

$$\frac{w_a}{V_s}(1,0)_{\delta} = \frac{(1-w_{x_d})}{\sin\theta_t}\sum_{n=1}^{\infty}\bar{S}_n \sin n\theta_t \tag{5.6-7}$$

where

$$\bar{S}_n = \frac{2}{\pi}\int_0^{\pi} S'(\theta)\,\bar{k}_2\,E(k_2)\sin\theta\cos n\theta\,d\theta$$

The axial velocity induced by the ring vortex at the propeller lifting line is given by equation (A-18) as

$$\frac{w_a}{V_s}(x,0,0)_{\gamma} = -\frac{h}{2\pi}\int_0^1\int_{-\pi}^{\pi}\frac{\left[\left(\frac{x}{x_d}\right)\cos\varphi'-1\right]\gamma(\varphi',z')\quad d\varphi'dz'}{\left[1+\left(\frac{x}{x_d}\right)^2+4h^2(a_t+z')^2-2\left(\frac{x}{x_d}\right)\cos\varphi'\right]^{3/2}} \tag{A-18}$$

If the Fourier series for γ is introduced, then this equation becomes

$$\frac{w_a}{V_s}(x,0,0)_{\gamma} = -\frac{h}{2\pi}\int_0^1\frac{k_2^3}{8\left(\frac{x}{x_d}\right)^{3/2}}\left(\sum_{n=0}^{\infty}g_n(z')\left[\int_{-\pi}^{\pi}\frac{\left[\left(\frac{x}{x_d}\right)\cos\varphi'-1\right]\cos n\varphi'd\varphi'}{(1-k_2^2\cos\frac{1}{2}\varphi')^{3/2}}\right)dz' \tag{5.6-8}$$

The function k_2 is given by equation (5.6-4) and the Fourier coefficient $h_n(z')$ of the ring vortex strength integrates out. By algebraic manipulation this equation is reduced to a function of elliptic and Riegel functions.

$$\frac{w_a}{V_s}(x,0,0)_{\gamma} = -\frac{h}{4\pi}\int_0^1\frac{k_2^3}{\left(\frac{x}{x_d}\right)^{3/2}}\left(\sum_{n=0}^{\infty}g_n(z')\left[\int_0^{\pi/2}\frac{\left[\left(\frac{x}{x_d}\right)\cos 2\bar{\varphi}-1\right]\cos 2n\bar{\varphi}\,d\bar{\varphi}}{(1-k_2^2\cos^2\bar{\varphi})^{3/2}}\right]\right)dz'$$

$$= -\frac{h}{2\pi}\left(\frac{x_d}{x}\right)^{1/2}\int_0^1 k_2\,g_0(z')\left[K(k_2)-E(k_2)-\frac{\left(\frac{x}{x_d}-1\right)E(k_2)}{4h^2(a_t+z')^2+\left(\frac{x}{x_d}-1\right)^2}\right]dz'$$

$$-\frac{h}{4\pi}\left(\frac{x_d}{x}\right)^{3/2}\int_0^1 \sum_{n=1}^{\infty} g_n(z')\left[-G_n(k_2)+\frac{1}{2}\left(\frac{x}{x_d}\right)\left(G_{n+1}(k_2)+G_{n-1}(k_2)\right)\right]dz'$$

(5.6-9)

The $G_n(k_2)$ are functions tabulated by Riegels[37] and are given by equation (5.3-35). There are a number of singularities appearing in the integrands when the axial coordinate $z' = -a_t$ and $x = x_d = 1$, but all are removable. As in the case of equation (5.6-3) this difficulty only occurs when the propeller has the same diameter as the duct ($x_d=1$), the axial coordinate is at the lifting line ($z'=-a_t$), and the velocity is desired at the point ($x=1$). In addition to the singularity of the integrands at this point, it must be remembered that $g_n(z')$ may also have a square root singularity at the leading edge of the duct. For this reason the change of variable $z' = \frac{1}{2}(1+\cos\theta')$ is introduced into the preceding equation and the function $g^*(\theta)$ obtained.

$$\frac{w_a}{V_s}(x,0,0)_\gamma = -\frac{h}{2\pi}\left(\frac{x_d}{x}\right)^{1/2}\int_0^{\pi} k_2\, g_0^*(\theta')\cos\tfrac{1}{2}\theta\left[K(k_2)-E(k_2)-\frac{\left(\frac{x}{x_d}-1\right)E(k_2)}{2h^2(\cos\theta'-\cos\theta_t)^2+\left(\frac{x}{x_d}-1\right)^2}\right]d\theta'$$

$$-\frac{h}{4\pi}\left(\frac{x_d}{x}\right)^{3/2}\int_0^{\pi} k_2^3\left(\sum_{n=1}^{\infty} g_n^*(\theta')\left[-G_n(k_2)+\frac{1}{2}\left(\frac{x}{x_d}\right)\left(G_{n+1}(k_2)+G_{n-1}(k_2)\right)\right]\right)\cos\tfrac{1}{2}\theta'\,d\theta' \qquad (5.6\text{-}10)$$

where

$$\cos\theta_t = -(2a_t+1)$$

$$k_2^2 = \frac{4\left(\frac{x}{x_d}\right)}{h^2(\cos\theta' - \cos\theta_t)^2 + \left(\frac{x}{x_d}+1\right)^2} \tag{5.6-11}$$

If $x_d = x = 1$, then equation (5.6-10) reduces to the following.

$$\frac{w_a}{V_s}(1,0,0)_\gamma = -\frac{h}{2\pi}\int_0^{\pi} \bar{k}_2 g_o^*(\theta')\left[K(\bar{k}_2) - E(\bar{k}_2)\right]\cos\tfrac{1}{2}\theta' d\theta'$$
$$-\frac{h}{2\pi}\int_0^{\pi} \bar{k}_2^3\left(\sum_{n=1}^{\infty} g_n^*(\theta')\left[-G_n(\bar{k}_2) + \tfrac{1}{2}G_{n+1}(\bar{k}_2) + \tfrac{1}{2}G_{n-1}(\bar{k}_2)\right]\cos\tfrac{1}{2}\theta'\right)d\theta'$$
$$-\tfrac{1}{2}\gamma(0) \tag{5.6-12}$$

where

$\gamma(0)$ is the ring vortex strength at $z = 0$

$$\bar{k}_2^2 = \frac{4}{h^2(\cos\theta' - \cos\theta_t)^2 + 4}$$

The ring vortex strength $\gamma(0)$ arises from the properties of vortex sheets[14] (see Appendix A). In this equation the integrand of the first integral has a logarithmic singularity at $k_2=1$ arising from the elliptic integral K. The integrand of the second integral also has a logarithmic singularity arising from the functions G_n at $k_2=1$. To show this, the expansion of the functions G_n near $k_2=1$, equation (5.3-36), is substituted into equation (5.6-12). The singularity of the type ($\lim_{k\to 1}\frac{1}{1-k^2}$) cancels out and a logarithmic singularity is left. The logarithmic singularity causes no difficulty since it is a removable singularity and integrates out. For convenience in numerical calculations the change in

variable

$$\cos\theta' - \cos\theta_t = \cos^3\theta$$

may be introduced into equation (5.6-12) to remove the singularities.

The tangential velocity induced by the ring vortex distribution at the propeller lifting line is given by equation (A-19). If the Fourier series for the circulation distribution is introduced i to this equation, it is obvious that the even series integrates out and only the odd series is left, consequently

$$w_t(x,0,0)_\gamma = -\frac{h}{2\pi}\left(\frac{x_d}{x}\right)^{3/2}\int_0^1 \frac{k_2^3}{2}\left(\sum_{n=1}^{\infty} h_n(z')\left[\int_0^{\pi}\frac{2h(a_t+z')\sin 2\bar{\varphi}\sin 2n\bar{\varphi}\,d\bar{\varphi}}{(1-k_2^2\cos^2\bar{\varphi})^{3/2}}\right]\right)dz'$$

$$= \frac{h}{8\pi}\left(\frac{x_d}{x}\right)^{3/2}\int_0^1 k_2^3\left[2h(a_t+z')\right]\left(\sum_{n=1}^{\infty} h_n(z')\left[\int_0^{\pi/2}\frac{\cos 2(n+1)\bar{\varphi} - \cos 2(n-1)\bar{\varphi}}{(1-k_2^2\cos^2\bar{\varphi})^{3/2}}d\bar{\varphi}\right]\right)dz'$$

$$= \frac{h}{8\pi}\left(\frac{x_d}{x}\right)^{3/2}\int_0^{\pi} k_2^3\left[2h(\cos\theta'-\cos\theta_t)\right]\cos\tfrac{1}{2}\theta'\left(\sum_{n=1}^{\infty} h_n^*(\theta')\left[G_{n-1}(k_2) - G_{n+1}(k_2)\right]\right)d\theta' \tag{5.6-13}$$

where

$$\cos\theta_t = -(2a_t + 1)$$

$$k_2^2 = \frac{4\left(\frac{x}{x_d}\right)}{h^2(\cos\theta'-\cos\theta_t)^2 + \left(\frac{x}{x_d}+1\right)^2}$$

h_n^* is given by equation (5.4-2)

The integrand of this integral does not have any singularities and is zero when $k_2 = 1$.

The axial velocity induced by the duct trailing vortex sheet at the propeller lifting line is given by equation (5.1-2) with $\varphi = 0$ and $z = 0$. Again introducing the Fourier series expansion for the circulation distribution this equation becomes

$$\left[\frac{w_a}{V_s}(x,0,0)\right]_{\frac{d\gamma}{d\varphi}} = \frac{h}{2\pi}\int_0^1 \sum_{n=1}^{\infty} n g_n(z)\left(\int_{-\pi}^{\pi} \cos n\varphi' \int_0^{\infty}\left[\frac{1-\left(\frac{x}{x_d}\right)\cos(\varphi'-\alpha)}{R^3}\right] d\alpha\, d\varphi'\right) dz'$$

$$-\frac{h}{2\pi}\int_0^1 \sum_{n=1}^{\infty} n h_n(z)\left(\int_{-\pi}^{\pi} \sin n\varphi' \int_0^{\infty}\left[\qquad\qquad\right] d\alpha\, d\varphi'\right) dz' \tag{5.6-14}$$

where

$$R^2 = 1 + \left(\frac{x}{x_d}\right)^2 - 2\left(\frac{x}{x_d}\right)^2 \cos(\varphi'-\alpha) + \left[2h(-a_t\ z') + \alpha \tan\beta_d\right]^2 \tag{5.6-15}$$

The integrand of the infinite integral is singular at the point $x = x_d$, $\alpha = \varphi' = \left[\frac{2h(a_t + z')}{\tan\beta_d}\right]$ and is only singular when the propeller is the same diameter as the duct. The infinite integrals occurring in this integral are almost the same as given by the propeller induction factor, equation (4.3-7). If $x \neq x_d$, the order of integration can be interchanged in this equation and the infinite integral can be reduced as in Appendix E.

$$\left[\frac{w_a}{v_i}(x,0,0)\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{2h}{\pi\tan^3\beta_d}\int_0^{\pi}\left(\cos\tfrac{1}{2}\theta\left[\sum_{n=1}^{\infty} n g_n^*(\theta)\bar{i}_{a_1} - \sum_{n=1}^{\infty} n h_n^*(\theta)\bar{i}_{a_2}\right]\right)d\theta$$

(5.6-16)

where

$$\bar{i}_{a_1} = \int_0^{\pi/2}\left[\left(1-\frac{x}{x_d}\cos 2\varphi\right)\frac{\cos 2n\varphi}{\sigma}\right]\left(n\cos n\zeta\, K_1(n\sigma) - \frac{n\pi}{2}\left[I_1(n\sigma) - \mathbf{L}_{-1}(n\sigma)\right]\sin n\zeta\right.$$

$$\left. + \frac{\zeta}{\sigma\sqrt{\zeta^2+\sigma^2}} + \frac{n}{\sigma}\int_0^{\zeta}\frac{\rho\sin n(\rho-\zeta)\,d\rho}{\sqrt{\sigma^2+\rho^2}}\right)d\varphi \qquad (5.6\text{-}17)$$

$$\bar{i}_{a_2} = \int_0^{\pi/2}\left[\left(1-\frac{x}{x_d}\cos 2\varphi\right)\frac{\cos 2n\varphi}{\sigma}\right]\left(n\sin n\zeta\, K(n\sigma) \perp \frac{n\pi}{2}\left[I_1(n\sigma) - \mathbf{L}_{-1}(n\sigma)\right]\cos n\zeta\right.$$

$$\left. - \frac{n}{\sigma}\int_0^{\zeta}\frac{\rho\cos n(\rho-\zeta)}{\sqrt{\sigma^2+\rho^2}}\,d\rho\right)d\varphi \qquad (5.6\text{-}18)$$

and

$$\zeta = -\frac{2h}{\tan\beta_d}(\cos\theta - \cos\theta_t)$$

$$\sigma^2 = \frac{1}{\tan^2\beta_d}\left[\left(1+\frac{x}{x_d}\right)^2 - 4\left(\frac{x}{x_d}\right)\cos^2\varphi\right]$$

$$\cos\theta_t = -(2a_t+1)$$

If the behavior of the double integral in equation (5.6-14) is considered at $x=x_d$, then it can be shown that $\bar{i}_{a1}$ and $\bar{i}_{a2}$ are valid even for $x=x_d$ since

$$\left(1-\frac{x}{x_d}\cos 2\varphi\right)_{x=x_d} = 2\sin^2\varphi$$

$$\sigma_{x=x_d} = \frac{2}{\tan\beta_d} \sin\varphi$$

and

$$\lim_{\varphi \to 0} (\sin\varphi)\, K_1\,(n\,\sigma) = \left(\frac{\tan\beta_d}{2n}\right)$$

The tangential velocity induced by the duct trailing vortex sheet at the propell : lifting line is given by equation (5.1-4). Introducing the Fourier series expansion for the circulation distribution this equation becomes

$$\left[\frac{w_t}{V_s}(x,0,0)\right]_{\frac{\partial\Gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^{\pi} \cos\tfrac{1}{2}\theta \left[\sum_{n=1}^{\infty} n g_n^*(\theta)\,\bar{i}_{t_1} + \sum_{n=1}^{\infty} n\, h_n^*(\theta)\,\bar{i}_{t_2}\right] d\theta \qquad (5.6\text{-}19)$$

where

$$\bar{i}_{t_1} = \int_{-\pi}^{\pi} \cos n\varphi \int_0^{\infty} \left[\frac{-2h(\cos\theta - \cos\theta_t)\sin(\varphi' - \alpha) + \left[\frac{x}{x_d} - \cos(\varphi' - \alpha) + \alpha\sin(\varphi' - \alpha)\right]\tan\beta_d}{R^3}\right] d\alpha\, d\varphi'$$

$$= -\frac{4}{\tan^2\beta_d}\int_0^{\pi/2} \frac{\cos 2n\varphi}{\sigma^2}\left[\left(\frac{x}{x_d}\right) - \cos 2\varphi\right]\left(\frac{\zeta}{\sqrt{\sigma^2+\zeta}} + n\int_0^{\zeta}\frac{\rho\sin n(\rho-\zeta)}{\sqrt{\sigma^2+\rho^2}}\,d\rho\right) d\varphi$$

$$+ \frac{4n}{\tan^2\beta_d}\int_0^{\pi/2} \sin 2n\varphi \sin 2\varphi\left(\int_0^{\zeta}\frac{\cos n(\rho-\zeta)}{\sqrt{\sigma^2+\rho^2}}\,d\rho\right) d\varphi$$

$$- \frac{4n}{\tan^2\beta_d}\int_0^{\pi/2} \frac{\cos 2n\varphi}{\sigma}\left[\left(\frac{x}{x_d}\right) - \cos 2\varphi\right]\left(\cos n\zeta\, K_1(n\sigma) - \frac{\pi}{2}\sin n\zeta\left[I_1(n\sigma) - \mathbf{L}_1(n\sigma)\right]\right) d\varphi$$

$$- \frac{4n}{\tan^2\beta_d}\int_0^{\pi/2} \sin 2n\varphi \sin 2\varphi\left(\cos n\zeta\, K_0(n\sigma) + \frac{\pi}{2}\sin n\zeta\left[I_0(n\sigma) - \mathbf{L}_0(n\sigma)\right]\right) d\varphi \qquad (5.6\text{-}20)$$

and

$$\bar{i}_{t_2} = \int_{-\pi}^{\pi} \sin n\varphi \int_{0}^{\infty} \left(\frac{-2h(\cos\theta - \cos\theta_t)\sin(\varphi' - \alpha) + \left[\frac{x}{x_d} - \cos(\varphi' - \alpha) + \alpha\sin(\varphi' - \alpha)\right]\tan\beta_d}{R^3} \right) d\alpha\, d\varphi'$$

$$= \frac{4n}{\tan^2\beta_d} \int_0^{\pi/2} \frac{\left(\frac{x}{x_d} - \cos 2\varphi\right)}{\sigma^2} \cos 2n\varphi \left(\int_0^{\varsigma} \frac{\rho\cos n(\rho - \varsigma)}{\sqrt{\sigma^2 + \rho^2}} d\rho \right) d\varphi$$

$$+ \frac{4}{\tan^2\beta_d} \int_0^{\pi/2} \sin 2\varphi \sin 2n\varphi \left(\frac{1}{\sqrt{\sigma^2 + \varsigma^2}} - n \int_0^{\varsigma} \frac{\sin n(\rho - \varsigma)}{\sqrt{\sigma^2 + \rho^2}} d\rho \right) d\varphi$$

$$- \frac{4n}{\tan^2\beta_d} \int_0^{\pi/2} \left(\frac{x}{x_d} - \cos 2\varphi\right) \frac{\cos 2n\varphi}{\sigma} \left| \sin n\varsigma\, K_1(n\sigma) + \frac{\pi}{2}\cos n\varsigma\left[I_1(n\sigma) - \mathbf{L}_{-1}(n\sigma)\right] \right) d\varphi$$

$$- \frac{4n}{\tan^2\beta_d} \int_0^{\pi/2} \sin 2\varphi \sin 2n\varphi \left| \sin n\varsigma\, K_0(n\sigma) - \frac{\pi}{2}\cos n\varsigma\left[I_0(n\sigma) - \mathbf{L}_0(n\sigma)\right] \right| d\varphi$$

(5.6-21)

where

$$\varsigma = -\frac{2h}{\tan\beta_d}(\cos\theta - \cos\theta_t)$$

$$\sigma^2 = \frac{1}{\tan^2\beta_d}\left[\left(1 + \frac{x}{x_d}\right)^2 - 4\left(\frac{x}{x_d}\right)\cos^2\varphi\right]$$

$$\cos\theta_t = -(2a_t + 1)$$

These last two equations have no singularities in the integrands of the integrals. There are some indeterminate forms of 0/0, which are evaluated as follows

If $\varphi = 0$

$$\lim_{x \to x_d} \left(\frac{\cos 2n\varphi}{\sigma^2} \left[\frac{x}{x_d} - \cos 2\varphi \right] \right) = \frac{\tan^2 \beta_d}{2}$$

$$\lim_{x \to x_d} \left(\frac{\sin 2n\varphi \sin 2\varphi}{\sigma^2} \right) = n \tan^2 \beta_d$$

If $x = x_d$

$$\lim_{\varphi \to 0} \left(\frac{\cos 2n\varphi}{\sigma} \left[1 - \cos 2\varphi \right] K_1(n\sigma) \right) = \frac{\tan^2 \beta_d}{2n}$$

$$\lim_{\varphi \to 0} \left(\frac{\sin 2n\varphi \sin 2\varphi}{\sigma} K_1(n\sigma) \right) = \tan^2 \beta_d$$

With the preceding equations the axial and tangential velocities induced by the duct can be determined at the propeller lifting line by equations (5.6-1) and (5.6-2). These equations allow the propeller to have any axial position in relation to the duct and, in addition, the diameter of either the duct or propeller is arbitrary.

In addition to the velocity induced by the duct at the propeller, the radial velocities induced at the hub are needed. Because of the way the hub problem was treated only the average radial velocity is desired. The average radial velocity induced by the propeller at the hub was essentially derived earlier in Section V.3 and is

$$\left[\frac{w_r}{V_s}(x_h, z) \right]_{\varphi} = \frac{4b}{\pi x_h} \int_{x_h}^{1} \frac{G_s'(x_o)}{\tan \beta_i} \left(\frac{x_o}{x_h} \right)^{\frac{1}{2}} \bar{i}_{r_o}\left(\frac{x_o}{x_h}, z \right) dx_o \tag{5.6-22}$$

The function $\bar{i}_{r_o}\left(\frac{x_o}{x_h}, z \right)$ is given by equation (5.3-12) with x_d replaced by x_h. The radial velocity induced at the hub by the ring source distribution is derived in Appendix B as equation (B-13) and will not be repeated here. The radial

velocity induced at the hub by the ring vortex distribution can be derived from equation (A-14). If the Fourier series for the ring vortex strength is substituted into this equation, the average radial velocity induced at the hub is found to be

$$\left[\frac{w_t}{V_s}(x_h,\theta)\right] = \frac{h}{4\pi}\left(\frac{x_d}{x_h}\right)^{3/2}\int_0^{\pi} g_o^*(\theta')\cos\tfrac{1}{2}\theta'\, k\left[2h(\cos\theta-\cos\theta')\right]\left(\frac{(2-k^2)E(k)}{1-k^2} - 2K(k)\right)d\theta' \tag{5.6-23}$$

where

$$\cos\theta = 2(z-a_t)-1$$

$$k^2 = \frac{4\left(\frac{x_h}{x_d}\right)}{4h^2(\cos\theta-\cos\theta')^2 + \left(1+\frac{x_h}{x_d}\right)^2}$$

The modulus k is always different than one so the integral is always regular. The average radial velocity induced by the trailing vortex system is obtained from equation (5.1-3). If the Fourier expansion for the circulation distribution is substituted into this equation and then the equation integrated with respect to φ from 0 to 2π, it is easy to show that the average radial velocity at the hub is zero, i.e.

$$\left[\frac{w_t}{V_s}(x_h,z)\right]_{\frac{\partial i}{\partial\varphi}} = 0 \tag{5.6-24}$$

Many of the coefficients derived in this section are dependent on geometry only so can be tabulated. Specifically the coefficients $\bar{I}_{a_1}$, $\bar{I}_{a_2}$, $\bar{I}_{t_1}$, and $\bar{I}_{t_2}$ should be mentioned.

VI. Conclusions

A theory for the ducted propeller is developed which can be used for design purposes. The method is based on the assumption of an inviscid fluid and that the propeller can be represented by a lifting line. These, among others, are the assumptions normally made in propeller theory[33] and, consequently, corrections must be introduced to allow for the difficulties in these assumptions. These corrections are not considered here.

Assuming that lifting surface and viscous corrections can be made, the adequacy of the theory must still be based on experimental results. A linearized theory similar to that used for treating the annular airfoil has been very useful in two-dimensional airfoil theory and, likewise, the treatment of the propeller by lifting line theory has been effective. It is not self-evident, however, that in combination the resulting theory of the ducted propeller will be satisfactory. It is presumed that it will be.

An attempt has been made not to restrict the problem more than the basic assumptions which are given in the introduction, consequently, somewhat cumbersome equations are obtained. In general, these equations have been reduced to coefficient form which are dependent only on geometry so can be tabulated.

Accepting the basic assumptions in Section I, many

important observations can be made about the annular airfoil and the ducted propeller.

1. The linearization of the duct boundary conditions results in the radial velocities induced by the singularities in the flow being equal the duct surface slope. In addition the boundary conditions are only satisfied on a cylinder of constant diameter and chord equal to the duct chord.

2. The strength of the duct source ring distribution is independent of angular position and dependent only on the duct thickness distribution even in the presence of the propeller.

3. The strength of the duct ring vortex distribution depends on both the camber and thickness distribution of the duct as well as the radial velocities induced by the propeller. If the duct is at zero incidence and the propeller is not present, the ring vortex distribution is independent of angular position and no free vortices are shed from the duct.

4. When the duct is at an angle of attack (no propeller present), the duct vortex strength is a sum of two terms. One term is the airfoil at zero incidence and the other is dependent on only the angle of attack, duct chord-diameter ratio, and the angular position.

5. The problem of the design of the propeller in the duct reduces essentially to the problem of the propeller by itself with the inclusion of the velocities induced by the

duct. This means the induction factors which have been calculated[33] are applicable to this problem.

6. On expanding the radial velocity induced by the propeller on the duct in a Fourier series, it is found that only terms exist which are harmonic with the number of blades except for one zero-order term. This means that the Fourier coefficients of duct vortex strength are harmonic with the number of blades except for one term. This zero-order term embodies the propeller average radial velocity and the duct thickness and camber. All the higher order terms of the duct vortex strength are functions of only the propeller induced velocity.

7. The duct ring source distribution induces no tangential velocity at the propeller nor does the source distribution representing the hub.

8. The duct trailing vortex system induces no average radial velocities at the hub.

9. The induced drag of the duct is zero if the duct is by itself and at zero incidence. In the presence of the propeller, the induced drag is dependent on the radial velocities induced by the hub, propeller and duct trailing vortex system.

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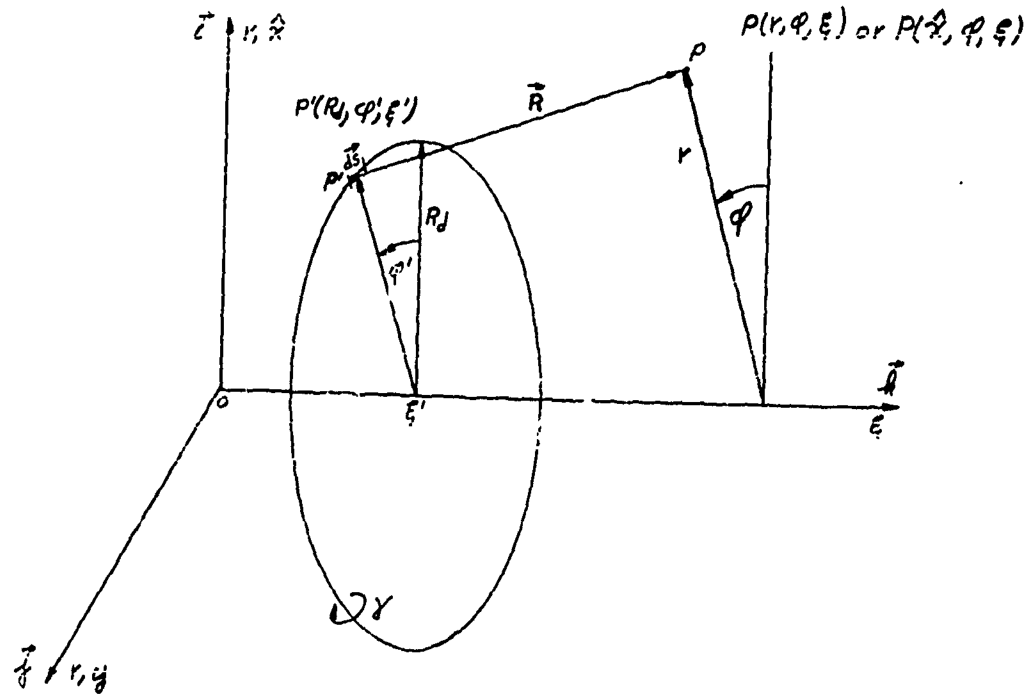
Appendix A

Velocity Induced by a Vortex Distribution on a Cylinder[13]

First consider the stream function and velocity distribution induced by a single vortex ring of diameter, R_d, located at ξ' . The figure shows such a ring where both a cylindrical coordinate system (r, φ, ξ) and a cartesian coordinate system $(\hat{x}, y, \xi)$ are used.



The coordinate ξ is in direction of the axis of the vortex, the element of the vortex filament is at the point, P', and the stream function and velocity distribution is desired

at the point, P. The Biot-Savort law gives the velocity induced by a vortex filament. In vector notation and for an element of filament $\vec{ds}$, this law is given by the following equation:

$$d\vec{V} = -\frac{\gamma(\varphi')}{4\pi}\frac{\vec{R}\times\vec{ds}}{R^3} \tag{A-1}$$

where

$d\vec{V}$ is the induced velocity at P from the element

$\vec{R}$ is the vector from point P' to P

$\vec{ds}$ is the incremented vector $\vec{\tau}ds$ tangent to the vortex ring at p'

$\gamma(\varphi)$ is the circulation at P and dependent on φ'

If the unit vectors $\vec{i},\vec{j},\vec{k}$ are in direction of the $\hat{x}$,y,ξ axes, the unit vector $\frac{\vec{ds}}{ds}$ can be written as

$$\frac{\vec{ds}}{ds} = \frac{R_d d\varphi'}{ds}\left[-\sin\varphi'\vec{i} + \cos\varphi'\vec{j} + 0\vec{k}\right] \tag{A-2}$$

Since $R_d d\varphi' = ds$ the above equation becomes

$$\frac{\vec{ds}}{ds} = -\sin\varphi'\vec{i} + \cos\varphi'\vec{j} + 0\vec{k} \tag{A-3}$$

From the figure it can be seen that the radius vector $\vec{R}$ from P' to P is

$$\vec{R} = (r\cos\varphi - R_d\cos\varphi')\vec{i} + (r\sin\varphi - R_d\sin\varphi')\vec{j} + (\xi - \xi')\vec{k} \tag{A-4}$$

By vector multiplication it follows that $\vec{R}\times\frac{\vec{ds}}{ds}$ is

$$\vec{R} \times \frac{d\vec{s}}{ds} = \left[(\xi - \xi')\cos\varphi'\right]\vec{i} + \left[(\xi - \xi')\sin\varphi'\right]\vec{j} + \left[R_d - r\cos(\varphi - \varphi')\right]\vec{k} \tag{A-5}$$

The magnitude of $\vec{R}$ is given by

$$R = \left|\vec{R}\right| = \sqrt{R_d^2 + r^2 + (\xi - \xi')^2 - 2rR_d\cos(\varphi - \varphi')} \tag{A-6}$$

Substituting equations (A-5) and (A-6) into equation (A-1) and integrating around the ring gives the velocity induced at P by a single vortex ring.

$$\left|\vec{V}_i\right|_{\substack{single\\ring}} = \frac{R_d}{4\pi}\int_0^{2\pi}\left(\frac{[(\xi-\xi')\cos\varphi']\vec{i} + [(\xi-\xi')\sin\varphi']\vec{j} - [r\cos(\varphi-\varphi') - R_d]\vec{k}}{[R_d^2 + r^2 + (\xi-\xi')^2 - 2rR_d\cos(\varphi-\varphi')]^{3/2}}\right)\gamma(\varphi')d\varphi' \tag{A-7}$$

For the induced velocity from a vortex cylinder, the induced velocity from the single vortex ring is integrated along the cylinder from the trailing edge ($\xi = a_t$) to the leading edge ($\xi = a_b$). Since each ring will not necessarily have the same strength, the vortex distribution will also be a function of ξ. It follows then that for the vortex cylinder the induced velocity is

$$\vec{V}_i(r,\varphi,\xi) = \frac{R_d}{4\pi}\int_{a_t}^{a_l}\int_0^{2\pi}\left(\frac{[(\xi-\xi')\cos\varphi']\vec{i} + [(\xi-\xi')\sin\varphi']\vec{j} - [r\cos(\varphi-\varphi') - R_d]\vec{k}}{[R_d^2 + r^2 + (\xi-\xi')^2 - 2rR_d\cos(\varphi-\varphi')]^{3/2}}\right)\cdot\gamma(\varphi',\xi')d\varphi'd\xi' \tag{A-8}$$

As expressed by equation (2.2-23) the axial coordinate (ξ) will be assumed to be nondimensionalized by the cylinder chord (a), the radial coordinate (r) by the propeller radius (R_p), the velocity and also the vortex distribution by the ship

speed or in the case of uniform flow, the free-stream velocity. The duct chord diameter ratio ($h = a/2R_d$) will also be introduced and then making the change of variable ($z'' = z' + a_t$) and noting that in the nondimensionalized form $a_\infty - a_t = 1$, the preceding equation can be written as

$$\frac{\vec{V}_i}{V_s}(x,\varphi,z)_\gamma = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{[2h(z-a_t-z')\cos\varphi']\vec{i} + [2h(z-a_t-z')\cos\varphi']\vec{j} - [\frac{x}{x_d}\cos(\varphi-\varphi')-1]\vec{k}}{[1+(\frac{x}{x_d})^2 + 4h^2(z-a_t-z')^2 - 2(\frac{x}{x_d})\cos(\varphi-\varphi')]^{3/2}}\right)\gamma(\varphi',z')d\varphi' dz' \tag{A-9}$$

If there is no clearance between the propeller and the duct, then $x_d = 1$ and the above equation becomes

$$\frac{\vec{V}_i}{V_s}(x,\varphi,z)_\gamma = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{[2h(z-a_t-z')\cos\varphi']\vec{i} + [2h(z-a_t-z')\cos\varphi']\vec{j} - [x\cos(\varphi-\varphi')-1]\vec{k}}{[1+x^2 + 4h^2(z-a_t-z')^2 - 2x\cos(\varphi-\varphi')]^{3/2}}\right)\gamma(\varphi',z')d\varphi' dz' \tag{A-10}$$

From the figure at the beginning of this section, it can be seen that the velocity in the axial direction (w_a) is given by the component in the $\vec{k}$ direction. The radial velocity component (w_r) and tangential velocity component (w_t) are obtained using the relationship of velocities in cylindrical and cartesian coordinates.[38] The velocity (w_x) is the component of velocity in the $\vec{i}$ direction and (w_y) is the component of velocity in the $\vec{j}$ direction, the radial and tangential velocities are given by the following equations

$$w_r = w_x\cos\varphi + w_y\sin\varphi$$

$$w_t = -w_x\sin\varphi + w_y\cos\varphi \tag{A-11}$$

Applying these relationships to equation (A-9) the different velocity components are found to be

$$\frac{w_a}{V_s}(x,\varphi,z)_\gamma = -\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\left[\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')-1\right]\gamma(\varphi',z')\,d\varphi'\,dz'}{\left[1+\left(\frac{x}{x_d}\right)^2+4h^2(z-a_t-z')^2-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}} \quad \text{(A-12)}$$

$$\frac{w_r}{V_s}(x,\varphi,z)_\gamma = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{2h(z-a_t-z')\cos(\varphi-\varphi')\,\gamma(\varphi',z')\,d\varphi'\,dz'}{\left[1+\left(\frac{x}{x_d}\right)^2+4h^2(z-a_t-z')^2-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}} \quad \text{(A-13)}$$

$$\frac{w_t}{V_s}(x,\varphi,z)_\gamma = -\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{2h(z-a_t-z')\sin(\varphi-\varphi')\,\gamma(\varphi',z')\,d\varphi'\,dz'}{\left[1+\left(\frac{x}{x_d}\right)^2+4h^2(z-a_t-z')^2-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}} \quad \text{(A-14)}$$

The velocity induced on the vortex cylinder itself is found by considering the properties of vortex sheets.[14] The velocity across a singular vortex sheet has a discontinuity in the tangential velocity component of magnitude γ. As a result the value of the induced velocity depends on the side of the sheet. Since only the axial and radial velocities will be needed on the ring itself, it follows then from vortex theory that

$$\frac{w_a}{V_s}(x_d\pm 0,\varphi,z)_\gamma = -\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\left[\cos(\varphi-\varphi')-1\right]\gamma(\varphi',z')\,d\varphi'\,dz'}{\left[4h^2(z-a_t-z')^2+2-2\cos(\varphi-\varphi')\right]^{3/2}} \pm \frac{1}{2}\gamma(\varphi,z) \quad \text{(A-15)}$$

$$\frac{w_r}{V_s}(x_d,\varphi,z)_\gamma = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{2h(z-a_t-z')\cos(\varphi-\varphi')\,\gamma(\varphi',z')\,d\varphi'\,dz'}{\left[4h^2(z-a_t-z')^2+2-2\cos(\varphi-\varphi')\right]^{3/2}}. \quad \text{(A-16)}$$

The plus sign refers to the outside of the duct and the minus sign to the inside. The radial velocity will also be needed on the propeller hub. Denoting the propeller hub radius

by x_h, this radial velocity is given by

$$\frac{w_r}{V_s}(x_h,\varphi,z)_\gamma = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{2h(z-a_t-z')\cos(\varphi-\varphi')\gamma(\varphi',z')\;d\varphi'dz'}{\left[4h^2(z-a_t-z')^2+1+\left(\frac{x_h}{x_d}\right)^2-2\left(\frac{x_h}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}} \tag{A-17}$$

The axial and tangential velocity induced at each propeller blade will also be needed. For this purpose it is sufficient to consider the propeller blade in its vertical position thus it is sufficient to consider the angle φ as zero and since z=o at the propeller, the axial and tangential velocities induced by the vortex cylinder at the propeller are

$$\frac{w_a}{V_s}(x,0,0)_\gamma = -\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\left[\left(\frac{x}{x_d}\right)\cos\varphi'-1\right]\gamma(\varphi',z')\;d\varphi'dz'}{\left[4h^2(a_t+z')^2+1+\left(\frac{x}{x_d}\right)^2-2\left(\frac{x}{x_d}\right)\cos\varphi'\right]^{3/2}} \tag{A-18}$$

$$\frac{w_t}{V_s}(x,0,0)_\gamma = -\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{2h(a_t+z')\sin\varphi'\gamma(\varphi',z')\;d\varphi'dz'}{\left[4h^2(a_t+z')^2+1+\left(\frac{x}{x_d}\right)^2-2\left(\frac{x}{x_d}\right)\cos\varphi'\right]^{3/2}} \tag{A-19}$$

If the circulation distribution $\gamma(\varphi, z)$ is a function of the angular position, then a vortex system is shed from the nozzle. The velocity induced by this system must also be considered and is discussed in Appendix C and D.

If the circulation distribution $\gamma(\varphi, z)$ is independent of the angle φ, then the velocity induced from point to point is independent of the angular position of the points. It is then sufficient to consider that the angle φ is zero. From the figure at the beginning of this section it can be seen that

for $\varphi = 0$ the radial velocity at P is given by the component of velocity in the $\vec{i}$ direction and the tangential velocity component is given by the component in the $\vec{j}$ direction. From considerations of symmetry it is obvious that the tangential component of velocity is zero. This also follows from the fact that the coefficient of $\vec{j}$ is an odd function and the integral of this term vanishes. Utilizing these results, equation (A-9) reduces to the following equation.

$$\left[\frac{\vec{V}_i(x,z)}{V_s}\right]_\gamma = \frac{h}{2\pi}\int_0^1 \frac{\gamma(z')\,k_1^2}{8\left(\frac{x}{x_d}\right)^{3/2}}\left(\int_0^{2\pi}\left[\frac{[2h(z-a_t-z')\cos\varphi']\vec{i}-\left[\frac{x}{x_d}\cos\varphi'-1\right]\vec{k}}{\left(1-k_1^2\cos^2\frac{1}{2}\varphi'\right)^{3/2}}\right]d\varphi'\right)dz' \tag{A-20}$$

where

$$k_1^2 = \frac{4\left(\frac{x}{x_d}\right)}{4h^2(z-a_t-z')^2+\left(1+\frac{x}{x_d}\right)^2}$$

The integral in the brackets can be reduced to the form of complete elliptic integrals by introducing the change in variable $\cos\frac{1}{2}\varphi' = \sin\theta$, then $d\varphi' = -2d\theta$ and when $\varphi'=0$, $\theta = \pi/2$ and when $\varphi' = 2\pi$, $\theta = \frac{3\pi}{2}$.

$$\left[\frac{\vec{V}_i(x,z)}{V_s}\right]_\gamma = \frac{h}{2\pi}\int_0^1 \frac{\gamma(z')\,k_1^3}{2\left(\frac{x}{x_d}\right)^{3/2}}\left(\int_0^{\pi/2}\left[\frac{[2h(z-a_t-z')(2\sin^2\theta-1)]\vec{i}-\left[2\left(\frac{x}{x_d}\right)\sin^2\theta-\left(\frac{x}{x_d}+1\right)\right]\vec{k}}{\left(1-k_1^2\sin^2\frac{1}{2}\theta\right)^{3/2}}\right]d\theta\right)dz'$$

$$= -\frac{h}{2\pi}\int_0^1 \frac{\gamma(z')\,k_1}{\left(\frac{x}{x_d}\right)^{1/2}}\left(\frac{2h(z-a_t-z')}{\left(\frac{x}{x_d}\right)}\left[K(k_1)-E(k_1)-\frac{2\left(\frac{x}{x_d}\right)E(k_1)}{4h^2(z-a_t-z')+\left(\frac{x}{x_d}-1\right)^2}\right]\vec{i}\right.$$

$$\left.+\left[K(k_1)-E(k_1)-\frac{\left(\frac{x}{x_d}-1\right)E(k_1)}{4h^2(z-a_t-z')^2+\left(\frac{x}{x_d}-1\right)^2}\right]\vec{k}\right)dz' \tag{A-21}$$

where

$$K(k_1) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k_1^2\sin^2\theta}}\,d\theta$$ = complete elliptic integral of the first kind

$$E(k_1) = \int_0^{\pi/2} \sqrt{1-k_1^2\sin^2\theta}\,d\theta$$ = complete elliptic integral of the second kind

The axial component of velocity is given by the coefficient of the unit vector $\vec{k}$ and the radial component by the coefficient of the unit vector $\vec{i}$.

As discussed previously the axial component of the induced velocity is discontinuous across the vortex cylinder therefore on the vortex cylinder itself, i.e. ($x = x_d$, $a_t \leqq z \leqq a_\ell$), the induced velocities are

$$\left[\frac{w_a}{V_s}(x_d,z)\right]_\gamma = -\frac{h}{2\pi}\int_0^1 \gamma(z')\,k\left[K(k) + E(k)\right]dz' \pm \frac{1}{2}\gamma(z) \tag{A-22}$$

$$\left[\frac{w_r}{V_s}(x_d,z)\right]_\gamma = -\frac{1}{4\pi}\oint_0^1 \frac{\gamma(z')\,k}{(z-a_t-z')}\left|4h^2(z-a_t-z')^2\left[K(k) - E(k)\right] - 2E(k)\right|dz' \tag{A-23}$$

where

$$k^2 = \frac{1}{h^2(z-a_t-z')^2+1}$$

Appendix B

Velocity Induced by a Source Distribution on a Cylinder

In the derivation of the velocity field from a source cylinder, first the velocity induced by a single source ring of diameter R_d located at ξ' will be considered. As for the vortex ring a cylindrical coordinate system (r, φ, ξ) and a cartesian coordinate system (ℓ, y, ξ) are considered as shown in Figure 8. The velocity induced by an element $\vec{ds}$ of the source ring of diameter R_d at the point P is[13]

$$d\vec{V} = \frac{q(\varphi', \xi')}{4\pi} \frac{\vec{R}}{R^3} ds \tag{B-1}$$

The strength of the source at the point (R_d, φ', ξ') is denoted by $q(\varphi', \xi')$. The vector $\vec{R}$ and its magnitude R have the same value as given by equations (A-4) and (A-6), therefore the induced velocity for a single ring is given by

$$\left[\vec{V}_i\right]_{\substack{single\\ring}} = \frac{R_d}{4\pi} \int_0^{2\pi} \left(\frac{(r\cos\varphi - R_d\cos\varphi')\vec{i} + (r\sin\varphi - R_d\sin\varphi')\vec{j} + (\xi - \xi')\vec{k}}{[R_d^2 + r^2 + (\xi - \xi')^2 - 2rR_d\cos(\varphi - \varphi')]^{3/2}} \right) q(\varphi', \xi')\, d\varphi' \tag{B-2}$$

The induced velocity is given by integrating single vortex rings along the cylinder from the trailing edge ($\xi = a_t$) to the leading edge ($\xi = a_\ell$). Since each ring will not necessarily have the same strength, the source distribution will also be a function of ξ. For a source cylinder it follows then that the induced velocity is

$$\left[\vec{V}_i(r, \varphi, \xi)\right]_q = \frac{R_d}{4\pi} \int_{a_t}^{a_\ell} \int_0^{2\pi} \left(\frac{(r\cos\varphi - R_d\cos\varphi')\vec{i} + (r\sin\varphi - R_d\sin\varphi')\vec{j} + (\xi - \xi')\vec{k}}{[R_d^2 + r^2 + (\xi - \xi')^2 - 2rR_d\cos(\varphi - \varphi')]^{3/2}} \right) q(\varphi', \xi')\, d\varphi'\, d\xi' \tag{B-3}$$

As for the vortex ring the axial component will be nondimensionalized by the cylinder chord (a) and the radius by the propeller diameter (R_p). The induced velocity and source strength will be assumed to be nondimensionalized by the ship speed or in the case of uniform flow by the free-stream velocity. The chord diameter ratio of the cylinder ($h = a/2R_d$) will be introduced and the change of variable ($z'' = z' + a_\ell$). Since in the nondimensionalized form ($a_\ell - a_t = 1$), the preceding equation can be written as

$$\left[\frac{\vec{v}_\ell}{V_s}(x,\varphi,z)\right] = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{\left[\left(\frac{x}{x_d}\right)\cos\varphi - \cos\varphi'\right]\vec{i} + \left[\left(\frac{x}{x_d}\right)\sin\varphi - \sin\varphi'\right]\vec{j} + 2h(z-a_t-z')\vec{k}}{\left[4h^2(z-a_t-z')^2 + 1 + \left(\frac{x}{x_d}\right)^2 - 2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}}\right) q(\varphi',z')\,d\varphi'dz' \tag{B-4}$$

The axial component of velocity is given by the component in the $\vec{k}$ direction and the radial and tangential velocities follow from equation (A-11)

$$\frac{w_a}{V_s}(x,\varphi,z)_q = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{2h(z-a_t-z')\,q(\varphi',z')}{\left[4h^2(z-a_t-z')^2 + 1 + \left(\frac{x}{x_d}\right)^2 - 2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}}\right) d\varphi'dz' \tag{B-5}$$

$$\frac{w_r}{V_s}(x,\varphi,z)_q = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{\left[\left(\frac{x}{x_d}\right) - \cos(\varphi-\varphi')\right] q(\varphi',z')}{\left[4h^2(z-a_t-z')^2 + 1 + \left(\frac{x}{x_d}\right)^2 - 2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}}\right) d\varphi'dz' \tag{B-6}$$

$$\frac{w_t}{V_s}(x,\varphi,z)_q = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{\sin(\varphi-\varphi')\,q(\varphi',z')}{\left[4h^2(z-a_t-z')^2 + 1 + \left(\frac{x}{x_d}\right)^2 - 2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]^{3/2}}\right) d\varphi'dz' \tag{B-7}$$

The velocities induced on the source cylinder itself are found by considering the properties of source surfaces.[14] In this case the tangential velocities are continuous across the surface but the normal velocities are discontinuous. Denoting the outside of the cylinder by a plus sign and the inside by a minus sign, it follows then that the radial velocity on the cylinder itself ($x = x_d, a_t \leqq z \leqq a_\ell$) is given by[38]

$$\frac{w_r}{V_s}(x_d,\varphi,z) = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{[1-\cos(\varphi-\varphi')]\,q(\varphi',z')}{[4h^2(z-a_t-z')^2+2-2\cos(\varphi-\varphi')]^{3/2}}\right)d\varphi'\,dz' \pm \frac{1}{2}q(\varphi,z) \qquad \text{(B-8)}$$

If the source distribution is considered a constant with respect to φ, then, since the flow is symmetrical, it is sufficient to consider that the angle φ is zero in equation (B-4). In this case the velocity component in the $\vec{j}$ direction is an odd function and integrates out. If the change in variable $\cos\frac{1}{2}\varphi' = \sin\theta$ is made, equation (B-4) can be reduced to the form of complete elliptic integrals.

$$\left[\frac{\vec{V}_i}{V_s}(x,\varphi,z)\right]_q = \frac{2h}{\pi}\int_0^1 \frac{q(z')k_1^3}{8\left(\frac{x}{x_d}\right)^{3/2}}\left[\int_0^{\pi/2}\left(\frac{\left[\left(\frac{x}{x_d}+1\right)-2\sin^2\theta\right]\vec{i}+[2h(z-a_t-z')]\vec{k}}{[1-k_1^2\sin^2\theta]^{3/2}}\right)d\theta'\right]dz'$$

$$= \frac{h}{2\pi}\int_0^1 \frac{q(z')k_1}{\left(\frac{x}{x_d}\right)^{1/2}}\left(\left[\frac{K(k_1)}{\left(\frac{x}{x_d}\right)} - \left(1 - \frac{2\left(\frac{x}{x_d}\right)\left(\frac{x}{x_d}-1\right)}{4h^2(z-a_t-z')^2+\left(\frac{x}{x_d}+1\right)^2}\right)\frac{E(k_1)}{\left(\frac{x}{x_d}\right)}\right]\vec{i}\right.$$

$$\left. - \left[\frac{2[2h(z-a_t-z')]E(k_1)}{4h^2(z-a_t-z')^2+\left(\frac{x}{x_d}-1\right)^2}\right]\vec{k}\right)dz' \qquad \text{(B-9)}$$

where

$$k_1^2 = \frac{4\left(\frac{x}{x_d}\right)}{k^2(z-a_t-z')^2+\left(\frac{x}{x_d}+1\right)^2}$$

$$K(k_1) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k_1^2 \sin^2\theta}}\, d\theta,$$ complete elliptic integral of the first kind

$$E(k_1) = \int \sqrt{1-k_1^2 \sin^2\theta}\, d\theta,$$ complete elliptic integral of the second kind

The axial component of velocity is denoted by the component in the $\vec{k}$ direction and radial component is the component in the $\vec{i}$ direction. Again considering the properties of source surfaces, the velocities induced on the cylinder itself $x=x_d$, $a_t \leqq z \leqq a_\ell$) are found to be as follows

$$\frac{W_a}{V_s}(x_d,z)_q = \frac{1}{2\pi} \oint_0^1 \frac{q(z')\, k\, E(k)}{(z-a_t-z')}\, dz' \qquad \text{(B-10)}$$

and

$$\frac{W_r}{V_s}(x_d \pm 0,z)_q = \frac{h}{2\pi} \int_0^1 q(z')\, k\left[K(k) - E(k)\right] dz' \pm \frac{1}{2} q(z) \qquad \text{(B-11)}$$

where

$$k^2 = \frac{1}{k^2(z-a_t-z')^2+1}$$

It is shown in section (II.3) that within the linearized theory the source strength is independent of the angular position even in the presence of a propeller. For this reason the velocity induced by the source cylinder at each propeller blade is obtained from equation (B-9) by letting z=0. Since only the

axial velocity is needed, it is the only one given.

$$\frac{w_a(x,0)_q}{V_s} = -\frac{h}{\pi}\int_0^1 \frac{q(z')\,k_2}{\left(\frac{x}{x_d}\right)^{1/2}}\left(\frac{2h(a_t+z')\,E(k_2)}{4h^2(a_t+z')^2+\left(\frac{x}{x_d}-1\right)^2}\right)dz' \tag{B-12}$$

where

$$k_2^2 = \frac{4\left(\frac{x}{x_d}\right)}{4h^2(a_t+z')^2+\left(\frac{x}{x_d}+1\right)^2}$$

Similarly, the radial velocity induced at the hub by the source cylinder is independent of angle and is obtained from equation (B-9) by letting $x=x_h$

$$\frac{w_r(x_h,z)_q}{V_s} = \frac{h}{2\pi}\int_0^1 \frac{q(z')\,k_3}{\left(\frac{x_h}{x_d}\right)^{1/2}}\left(K(k_3)-E(k_3)+\frac{2\left(\frac{x_h}{x_d}\right)\left(\frac{x_h}{x_d}-1\right)E(k_3)}{4h^2(z-a_t-z')^2+\left(\frac{x_h}{x_d}-1\right)^2}\right)dz' \tag{B-13}$$

where

$$k_3^2 = \frac{4\left(\frac{x_h}{x_d}\right)}{4h^2(z-a_t-z')^2+\left(\frac{x_h}{x_d}+1\right)^2}$$

Appendix C

Velocities Induced by the Free Vortex System of a Vortex Cylinder at an Angle of Attack

If the ring vortices making up the vortex cylinder are a function of angular as well as axial position, then a system of free vortices will be shed from the cylinder. This is equivalent to the vortex sheet which is shed from a three dimensional flat wing.[27] Consequently similar assumptions will be made about the shape of the vortex sheet, namely, the free vortices form a cylinder which has a constant diameter equal to the diameter of the bound vortex cylinder with generators which are parallel to the ξ -axis, and which extends from a bound ring vortex to minus infinity. The density of the free vortices is $\frac{1}{R_d}\frac{\partial\gamma}{\partial\varphi}(\varphi, \xi)$, where $\gamma(\varphi, \xi)$ is the density of the bound vortices. When the annular airfoil is at angle of attack, the free vortex lines are straight lines parallel to the ξ -axis while in presence of a propeller the vortex lines are helical in shape. In this section only the straight vortex lines will be considered.

The same coordinate system and notation will be used as in Appendix A.

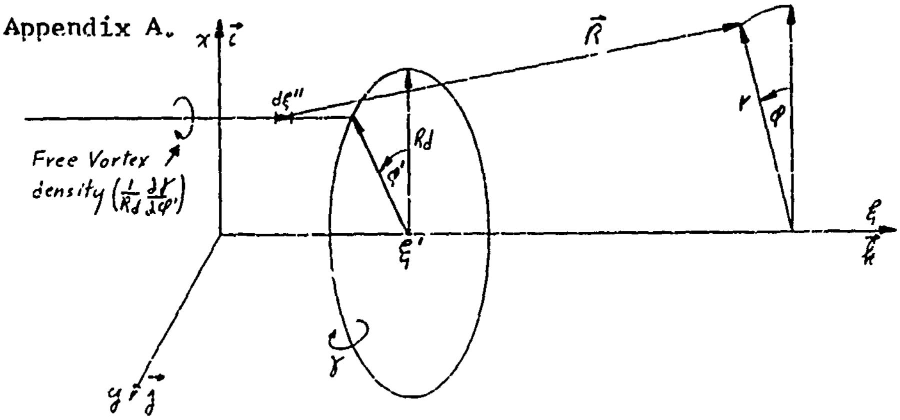


Figure 9. Notation for vortex cylinder trailing vortex system

The velocity induced by an element of the free vortex follows from the law of Biot-Savart, equation (A-1)

$$d\vec{V}_i = -\frac{1}{4\pi R_d}\frac{\partial\gamma}{\partial\varphi'}\frac{\vec{R}\times\vec{ds}}{R^3}d\xi' R_d d\varphi' \tag{C-1}$$

From the figure the vector $\vec{ds}$ is given by

$$ds = o\vec{i} + o\vec{j} + d\xi''\vec{k} \tag{C-2}$$

and the radius vector $\vec{R}$ from P' to P is

$$\vec{R} = (r\cos\varphi - R_d\cos\varphi')\vec{i} + (r\sin\varphi - R_d\sin\varphi')\vec{j} + (\xi - \xi'')\vec{k} \tag{C-3}$$

The magnitude of R is

$$R = |\vec{R}| = \sqrt{(\xi - \xi'')^2 + r^2 + R_d^2 - 2rR_d(\varphi - \varphi')} \tag{C-4}$$

From vector multiplication of equation (C-2) and (C-3), RXds is obtained

$$\vec{R}\times\vec{ds} = (r\sin\varphi - R_d\sin\varphi')\vec{i}\,d\xi'' - (r\cos\varphi - R_d\cos\varphi')\vec{j}\,d\xi'' \tag{C-5}$$

Substituting into equation (C-1) the velocity induced at P by an element of the free vortex is

$$d\vec{V}_i = -\frac{1}{4\pi R_d}\frac{\partial\gamma'}{\partial\varphi'}\left(\frac{(r\sin\varphi - R_d\sin\varphi')\vec{i} - (r\cos\varphi - R_d\cos\varphi')\vec{j}}{[(\xi - \xi'')^2 + r^2 + R_d^2 - 2rR_d\cos(\varphi - \varphi')]^{3/2}}\right)d\xi'' d\xi' R_d d\varphi' \tag{C-6}$$

Since the free vortex extends from $-\infty$ to ξ', the velocity induced by an infinitesimal strip is obtained by integration.

$$\left[\vec{V}_i\right]_{\substack{single\\ filament}} = -\frac{1}{4\pi R_d}\frac{\partial\gamma}{\partial\varphi'}\left[(r\sin\varphi - R_d\sin\varphi')\vec{i} - (r\cos\varphi - R_d\cos\varphi')\vec{j}\right]d\xi' R_d d\varphi'$$

$$\cdot\int_{-\infty}^{\xi'}\frac{d\xi''}{[(\xi - \xi'')^2 + r^2 + R_d^2 - 2rR_d\cos(\varphi - \varphi')]^{3/2}} \tag{C-7}$$

The integral is evaluated and equation (C-7) becomes

$$\left[\vec{V}_i\right]_{\substack{single\\filament}} = -\frac{1}{4\pi R_d}\frac{\partial \gamma}{\partial \varphi'}\left[(r\sin\varphi - R_d\sin\varphi')\vec{i} - (r\cos\varphi - R_d\cos\varphi')\vec{j}\right]$$

$$\cdot\left(\frac{1}{[r^2 + R_d^2 - 2rR_d\cos(\varphi-\varphi')]}\left[\frac{(\xi-\xi')}{\sqrt{(\xi-\xi')^2 + r^2 + R_d^2 - 2rR_d\cos(\varphi-\varphi')}} - 1\right]\right)d\xi' R_d\, d\varphi' \quad \text{(C-8)}$$

The velocity induced by all the free vortex filaments from a single vortex ring is given by integrating this equation with respect to φ' from 0 to 2π and of the free vortex cylinder by integrating the induced velocity of the single rings which are distributed along the chord of the cylinder.

$$\left[\vec{V}_i\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{1}{4\pi}\int_{a_t}^{a_\ell}\int_0^{2\pi}\frac{\left[(r\sin\varphi - R_d\sin\varphi')\vec{i} - (r\cos\varphi - R_d\cos\varphi')\vec{j}\right]}{[r^2 + R_d^2 - 2rR_d\cos(\varphi-\varphi')]}\left[\frac{(\xi-\xi')}{\sqrt{(\xi-\xi')^2 + r^2 + R_d^2 - 2rR_d\cos(\varphi-\varphi')}} + 1\right]\frac{\partial\gamma}{\partial\varphi'}\,d\varphi' d\xi' \quad \text{(C-9)}$$

Nondimensionalizing this equation as before, i.e. the axial component with the bound vortex cylinder length (a) the radial component with the ring radius (R_d) and the circulation with the velocity (V_o) the above equation is given more simply as:

$$\left[\frac{\vec{V}_i}{V_o}\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\left[(\frac{x}{R_d}\sin\varphi - \sin\varphi')\vec{i} - (\frac{x}{R_d}\cos\varphi - \cos\varphi')\vec{j}\right]}{\left[(\frac{x}{R_d})^2 + 1 - 2\frac{x}{R_d}\cos(\varphi-\varphi')\right]}\left[\frac{2h(\bar{z}-z')}{\sqrt{4h^2(\bar{z}-z')^2 + (\frac{x}{R_d})^2 + 1 - 2\frac{x}{R_d}\cos(\varphi-\varphi')}} + 1\right]\frac{\partial\gamma}{\partial\varphi'}\,d\varphi' dz'$$

where $\bar{z} = z - a_t$, z refers to the coordinate system at the propeller.

From the form of this equation it can be seen that the free vortex sheet induces no axial component of velocity. The

radial and tangential components are given by the usual relations of velocities in cartesian and polar coordinates, equation (A-11),

$$\left[\frac{W_r}{V_0}(x,\varphi,\bar{z})\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{\pi}\int_0^1\int_0^{2\pi}\frac{\sin(\varphi-\varphi')}{\left[\left(\frac{x}{x_d}\right)^2+1-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]}\left[\frac{2h(\bar{z}-z')}{\sqrt{4h^2(\bar{z}-z')^2+\left(\frac{x}{x_d}\right)^2+1-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')}}+1\right]\frac{\partial\gamma}{\partial\varphi'}d\varphi' dz' \tag{C-10}$$

and

$$\left[\frac{W_t}{V_0}(x,\varphi,\bar{z})\right]_{\frac{\partial\gamma}{\partial\varphi}} = -\frac{h}{2\pi}\int_0^1\int_0^{2\pi}\frac{\left[\frac{x}{x_d}-\cos(\varphi-\varphi')\right]}{\left[\left(\frac{x}{x_d}\right)^2+1-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')\right]}\left[\frac{2h(\bar{z}-z')}{\sqrt{4h^2(\bar{z}-z')^2+\left(\frac{x}{x_d}\right)^2+1-2\left(\frac{x}{x_d}\right)\cos(\varphi-\varphi')}}+1\right]\frac{\partial\gamma}{\partial\varphi'}d\varphi' dz' \tag{C-11}$$

On bound vortex cylinder $\frac{x}{x_d} = 1$ and $0 \leqq \bar{z} \leqq 1$, the radial velocity, equation (C-10), reduces to the following

$$\left[\frac{W_r}{V_0}(R_d,\varphi,\bar{z})\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{2\pi}\int_0^1\int_0^{2\pi}\left[\cot\tfrac{1}{2}(\varphi-\varphi')\right]\left[\frac{2h(\bar{z}-z')}{\sqrt{4h^2(\bar{z}-z')^2+4\sin^2\frac{1}{2}(\varphi-\varphi')}}+1\right]\frac{\partial\gamma}{\partial\varphi'}d\varphi' dz' \tag{C-12}$$

and introducing the notation

$$k^2 = \frac{1}{h^2(\bar{z}-z')^2+1}$$

equation (C-12) becomes

$$\left[\frac{W_r}{V_0}(R_d,\varphi,\bar{z})\right]_{\frac{\partial\gamma}{\partial\varphi}} = \frac{h}{4\pi}\int_0^1\int_0^{2\pi}\left[\cot\tfrac{1}{2}(\varphi-\varphi')\right]\left[\frac{k h(\bar{z}-z')}{\sqrt{1-k^2\cos^2\frac{1}{2}(\varphi-\varphi')}}+1\right]\frac{\partial\gamma}{\partial\varphi'}d\varphi' dz' \tag{C-13}$$

Appendix D

Velocity Induced by a Helical Shaped Vortex Line

The velocity component induced at a point in space by a vortex line is given by the law of Biot-Savart. In vector notation and for an element of filament $d\vec{s}$ of constant strength this law is given by equation (A-1) rewritten here as (D-1).

$$d\vec{V}_i = -\frac{\gamma}{4\pi}\,\frac{\vec{R}\times d\vec{s}}{R^3} \tag{D-1}$$

where

- $d\vec{V}_i$ is the induced velocity at a point P
- $\vec{R}$ is the radius vector from the vortex element to the point P
- $d\vec{s}$ is the vector tangent to the vortex ring at the element
- γ is the circulation along the helix and is constant

The following figure shows such a helical vortex line leaving from a point P', which will be taken as a point on the duct or the propeller blade, where both a cylindrical coordinate system (r, φ, ξ) and a cartesian coordinate system (x, y, ξ) are used. The point $P(x', y', \xi')$ is a general point in space and the point $P''(x'', y'', \xi'')$ lays on the vortex line.

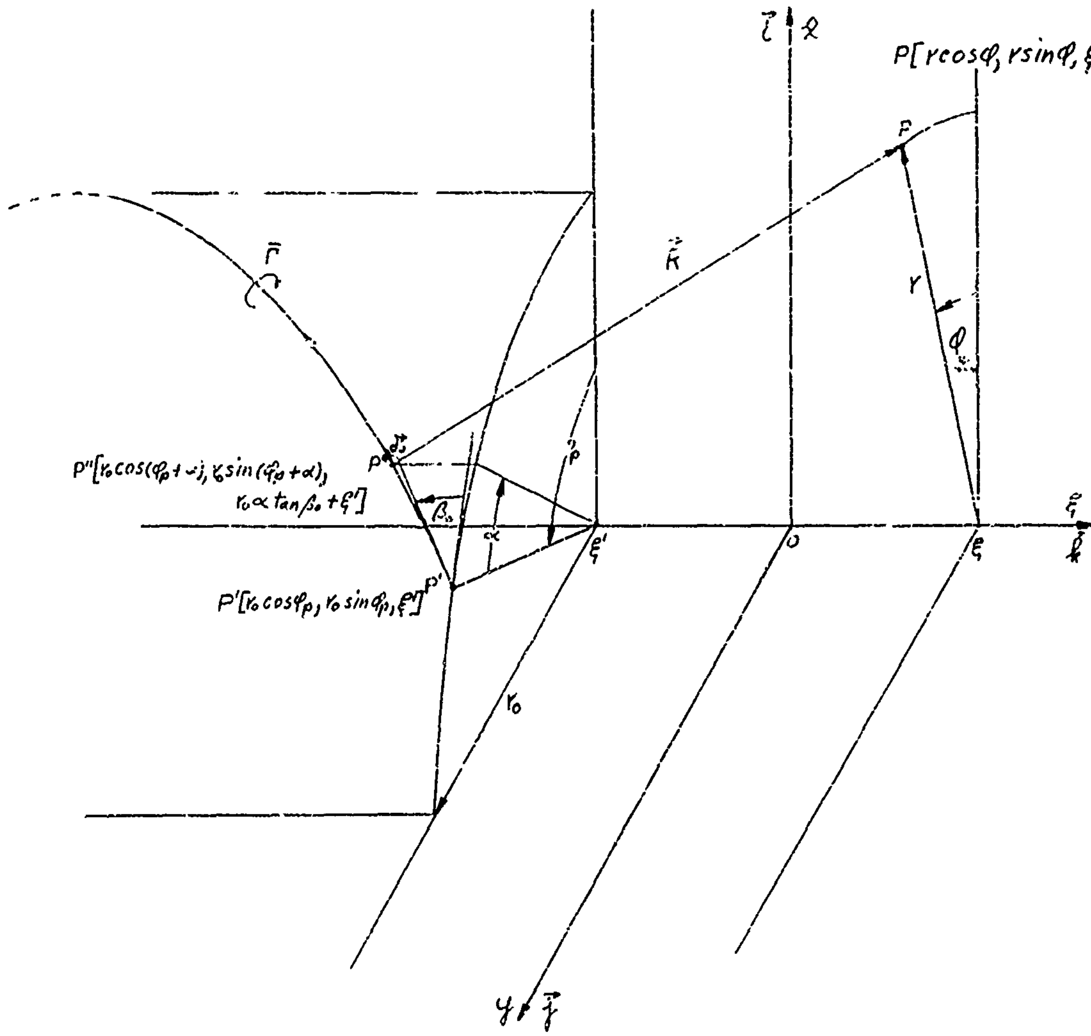


Figure 9. Notation of the helicoidal vortex line

From this figure it can be seen that the vector $\vec{R}$ is given by

$$\vec{R} = [r\cos\varphi - r_0\cos(\varphi_p + \alpha)]\vec{i} + [r\sin\varphi - r_0\sin(\varphi_p + \alpha)]\vec{j}$$

$$+ [\xi - \xi' - r_0\alpha\tan\beta_0]\vec{k} \qquad \text{(D-2)}$$

The magnitude of $\vec{R}$ is

$$R = |\vec{R}| = \left([r\cos\varphi - r_0\cos(\varphi_p+\alpha)]^2 + [r\sin\varphi - r_0\sin(\varphi_p+\alpha)]^2 + [\xi - \xi' - r_0\alpha\tan\beta_0]^2\right)^{1/2}$$

or

$$R^2 = r^2 + r_0^2 + [\xi - \xi' - r_0\alpha\tan\beta_0]^2 - 2rr_0\cos(\varphi - \varphi_p - \alpha)$$

(D-3)

The unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are in the direction of the x, y, ξ axes. The unit vector $\frac{d\vec{s}}{ds}$ also follows from the figure as

$$\frac{d\vec{s}}{ds} = \frac{-\sin(\varphi_p+\alpha)\vec{i} + \cos(\varphi_p+\alpha)\vec{j} + \tan\beta_0\vec{k}}{\sec\beta_0}$$

(D-4)

and $ds = r_0 \sec\beta_0 d\alpha$

The contribution of the whole vortex filament is found by integrating equation (D-1) with respect to α from 0 to $-\infty$

$$\vec{V}_i = -\frac{\Gamma}{4\pi}\int_0^{-\infty}\frac{\vec{R}\times d\vec{s}}{R^3} = \frac{\Gamma}{4\pi}\int_{-\infty}^{0}\frac{\vec{R}\times d\vec{s}}{R^3}$$

(D-5)

Substituting in for $\vec{R}$, equation (D-2) and $d\vec{s}$, equation (D-5), the induced velocity from a helical vortex line is given as

$$\vec{V}_i = \frac{\Gamma r_0}{4\pi}\int_{-\infty}^{0}\left[\left(\frac{[(\xi-\xi') - \alpha r_0\tan\beta_0]\cos(\varphi_p+\alpha) - [r\sin\varphi - r_0\sin(\varphi_p+\alpha)]\tan\beta_0}{R^3}\right)\vec{i}\right.$$

$$+\left(\frac{[(\xi-\xi')-\alpha r_o\tan\beta_o]\sin(\varphi_p+\alpha)+[r\cos\varphi-r_o\cos(\varphi_p+\alpha)]\tan\beta_o}{R^3}\right)\vec{j}$$

$$+\left(\frac{r_o-r\cos(\varphi-\varphi_p-\alpha)}{R^3}\right)\vec{k}\Bigg]d\alpha \quad ; \ (0<\tan\beta_o<\frac{\pi}{2})$$

(D-6)

The axial velocity induced by a helical vortex line is given by the component of velocity in the $\vec{k}$ direction and the radial and tangential velocities follow by using equations (A-11)

$$w_a=\frac{\Gamma r_o}{4\pi}\int_{-\infty}^{0}\left(\frac{r_o-r\cos(\varphi-\varphi_p-\alpha)}{R^3}\right)d\alpha \tag{D-7}$$

$$w_r=\frac{\Gamma r_o}{4\pi}\int_{-\infty}^{0}\left(\frac{[(\xi-\xi')-\alpha r_o\tan\beta_o]\cos(\varphi-\varphi_p-\alpha)-r_o\tan\beta_o\sin(\varphi-\varphi_p-\alpha)}{R^3}\right)d\alpha \tag{D-8}$$

$$w_t=\frac{\Gamma r_o}{4\pi}\int_{-\infty}^{0}\left(\frac{[r_o\alpha\tan\beta_o-(\xi-\xi')]\sin(\varphi-\varphi_p-\alpha)+[r-r_o\cos(\varphi-\varphi_p-\alpha)]\tan\beta_o}{R^3}\right)d\alpha \tag{D-9}$$

The integrands of the integrals for the axial and tangential velocities are singular when the point $P(r, \varphi, \xi)$ lays on the helical vortex line itself. This occurs when $\alpha = (\varphi - \varphi_p) = \frac{(\xi - \xi')}{r_o \tan\beta_o}$ and $r = r_o$. It can be shown that the integrand of the integral for the radial velocity

equation (D-8) is not singular at this point but has a jump discontinuity, except possibly at $\alpha = 0$. This is shown by letting $(\xi - \xi') = r_o(\varphi - \varphi_p)\tan\beta_o$ and $r = r_o$ in this equation, using the series expansion of the sine and cosine, and taking the limit.

$$\left(\frac{r_o[(\varphi-\varphi_p-\alpha]\tan\beta_o\cos(\varphi-\varphi_p-\alpha) - r_o\tan\beta_o\sin(\varphi-\varphi_p-\alpha)}{[2r_o^2 + r_o^2(\varphi-\varphi_p+\alpha)^2\tan^2\beta_o - 2r_o^2\cos(\varphi-\varphi_p-\alpha)]^{3/2}}\right)$$

$$= \frac{1}{r_o^2}\left(\frac{(\varphi-\varphi_p-\alpha)\tan\beta_o\cos(\varphi-\varphi_p-\alpha) - \tan\beta_o\sin(\varphi-\varphi_p-\alpha)}{\{2[1-\cos(\varphi-\varphi_p-\alpha)] + (\varphi-\varphi_p-\alpha)^2\tan^2\beta_o\}^{3/2}}\right)$$

The limit of this equation as $\alpha \rightarrow (\varphi - \varphi_p)$ is equivalent to considering $(\varphi - \varphi_p - \alpha) = \theta$ and letting $\theta \rightarrow 0$. Considering the series expansion of the sine and cosine, i.e.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

The above equation can be written as follows

$$\frac{1}{r_o^2}\left(\frac{\left(\theta - \frac{\theta^3}{2!} + \frac{\theta^5}{4!} - \cdots - \theta + \frac{\theta^3}{3!} \quad \frac{\theta^5}{5!}\cdots\right)\tan\beta_o}{\left[2\left(1 - 1 + \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right) + \theta^2\tan^2\beta_o\right]^{3/2}}\right)$$

$$= \frac{1}{r_o^2}\left(\frac{\left[-\frac{1}{3} + \left(\frac{1}{4!} - \frac{1}{5!}\right)\theta^2 + \cdots\right]\tan\beta_o}{\left[1 - 2\frac{\theta^2}{4!} + \qquad + \tan^2\beta_o\right]^{3/2}}\right)$$

and the limit of the integrand at $\theta \rightarrow 0$ is

$$\frac{\tan\beta_0}{r_0^2} \lim_{\theta \rightarrow 0} \left(\frac{-\frac{1}{3} + \left(\frac{1}{7!} \cdot \frac{1}{5!}\right)\theta^2 +}{\left[1 - 2\frac{\theta^2}{4!} + \cdots + \tan^2\beta_0\right]^{3/2}} \right)$$

$$= \frac{-\tan\beta_0}{3r_0^2\left[1 + \tan^2\beta_0\right]^{3/2}}$$

Since the integrand is a odd function of θ, the limit as $-\theta \rightarrow 0$ must be of opposite sign. Considering θ as equal $(\varphi - \varphi_p - \alpha)$ the limit of the integrand can then be written as

$$\lim_{\theta \rightarrow \pm 0} \left(\frac{1}{r_0^2} \left[\frac{(-\sin\theta + \theta\cos\theta)\tan\beta_0}{\left[2(1-\cos\theta) + \theta^2\tan^2\beta_0\right]^{3/2}} \right] \right) = \pm \frac{\tan\beta_0}{3r_0^2\left[1 + \tan^2\beta_0\right]^{3/2}} \quad \text{(D-10)}$$

Since the integrand of the integral for the radial velocity is not singular, it can easily be shown that this infinite integral for the radial velocity is uniformly convergent with respect to r_0, r, $\xi - \xi'$ and $\varphi - \varphi_p$ with the exceptions discussed in the following paragraph.

At the point $r = r_0, \xi = \xi', \varphi = \varphi_p$ and $\alpha = 0$, the integrand is discontinuous. If $\xi = \xi'$, $\varphi = \varphi_p$ and the limit of the integrand is taken as $\alpha \rightarrow 0$, obviously the limiting value is given by equation (D-10). If, however, $\varphi = \varphi_p$, $\alpha = 0$ and the limit is taken as $\xi \rightarrow \xi'$, or if $\xi = \xi'$, $\alpha = 0$ and the limit is taken as $\varphi \rightarrow \varphi_p$, the integrand is singular as $\frac{1}{0^2}$. At this point the integral does not exist and consequently is not uniformly convergent.

Appendix E

Evaluation of the Infinite Integrals in the Equation for the Velocity Induced by the Trailing Vortex System

In Section V.3 infinite definite integrals of the following two types arise

$$I_1 = \int_{-\pi}^{\pi} \sin n\bar{\theta} \left[\int_0^{\infty} \left(\frac{\left[\frac{a}{x}(\bar{z}-z') + \alpha\left(\frac{x_0}{x}\right)\tan\beta_0\right]\cos(\bar{\theta}+\alpha) - \left(\frac{x_0}{x}\right)\tan\beta_0 \sin(\bar{\theta}+\alpha)}{\left\{1 + \left(\frac{x_0}{x}\right)^2 + \left[\frac{a}{x}(\bar{z}-z') + \alpha\frac{x_0}{x}\tan\beta_0\right]^2 - 2\left(\frac{x_0}{x}\right)\cos(\bar{\theta}+\alpha)\right\}^{3/2}} \right) d\alpha \right] d\bar{\theta} \tag{E-1}$$

and

$$I_2 = \int_{-\pi}^{\pi} \cos n\bar{\theta} \left[\int_0^{\infty} \left(\frac{\left[\frac{a}{x}(\bar{z}-z') + \alpha\left(\frac{x_0}{x}\right)\tan\beta_0\right]\cos(\bar{\theta}+\alpha) - \left(\frac{x_0}{x}\right)\tan\beta_0 \sin(\bar{\theta}+\alpha)}{\left\{1 + \left(\frac{x_0}{x}\right)^2 + \left[\frac{a}{x}(\bar{z}-z') + \alpha\frac{x_0}{x}\tan\beta_0\right]^2 - 2\left(\frac{x_0}{x}\right)\cos(\bar{\theta}+\alpha)\right\}^{3/2}} \right) d\alpha \right] d\bar{\theta} \tag{E-2}$$

where

In Appendix D the infinite integral was shown to be uniformly convergent if $\bar{z} \neq z'$ and so the order of integration can be interchanged. If $\bar{z} = z'$, it can be shown that the asymptotic behavior of integral I_1 is finite while I_2 is infinite as $\lim_{x\to\infty} |\ln x|$. The integrals $j_{nc}^{(1)}$, equation (5.3-28), $j_{ns}^{(1)}$, equation (5.3-30), $i_n^{(c)}$, equation (5.3-41), and $i_n^{(s)}$, equation (5.3-42), are special forms of one or the other of the above integrals.

For simplification the following notation will be introduced:

$$\begin{aligned}
\hat{z} &= (\bar{z} - z')\frac{a}{x} \\
\bar{x} &= \frac{x_o}{x} \\
\beta &= \bar{x}\tan\beta_o \\
\zeta &= \frac{\hat{z}}{\bar{x}\tan\beta_o} \qquad\qquad \text{(E-3)} \\
\sigma^2 &= \frac{1}{\bar{x}^2\tan^2\beta_o}\left[(1+\bar{x})^2 - 4\bar{x}\cos^2\phi\right]
\end{aligned}$$

The integral I_1 will be discussed first. Interchange the order of integration and make the change of variable $\phi = (1/2)(\bar{\theta} + \alpha)$ in equation (E-1), then

$$I_1 = 2\int_0^{\infty}\int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}+\alpha}\left(\sin 2n\phi\cos n\alpha - \cos 2n\phi\sin n\alpha\right)\left(\frac{(\hat{z}+\alpha\beta)\cos 2\phi - \beta\sin 2\phi}{[(\hat{z}+\alpha\beta)^2 + (1+\bar{x})^2 - 4\cos^2\phi]^{3/2}}\right)d\alpha\, d\phi$$

It makes no difference whether the above integration is carried out over the range $(-\frac{\pi}{2}+\alpha \leq \phi \leq \frac{\pi}{2}+\alpha)$ or $(-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2})$ since the integrand is a periodic function of ϕ. Also, the part of the integrand which is an odd function of ϕ, integrates out, consequently the above integral can be written as

$$I_1 = -\frac{2}{\beta^2}\int_0^{\infty}\int_{-\pi/2}^{\pi/2}\left(\frac{(\zeta+\alpha)\cos 2\phi\cos 2n\phi\sin n\alpha + \sin 2\phi\sin 2n\phi\cos n\alpha}{[(\zeta+\alpha)^2 + \sigma^2]^{3/2}}\right)d\phi\, d\alpha$$

Interchange the order of integration again and make the change in variable $\rho = \zeta + \alpha$, then

$$I_1 = -\frac{2}{\beta^2}\int_{-\pi/2}^{\pi/2}\left(\cos 2\phi \cos 2n\phi\left[\int_0^{\infty}\frac{\rho \sin n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\,d\rho - \int_0^{\zeta}\frac{\rho \sin n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\,d\rho\right]\right)d\phi$$

$$-\frac{2}{\beta^2}\int_{-\pi/2}^{\pi/2}\left(\sin 2\phi \sin 2n\phi\left[\int_0^{\infty}\frac{\cos n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\,d\rho - \int_0^{\zeta}\frac{\cos n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\,d\rho\right]\right)d\phi$$

(E-4)

Assuming that the infinite integrals can be evaluated in the form of known functions, this equation is much easier to evaluate than equation (E-1) since the range of ζ, which is of interest, is restricted. The difficulty in evaluating numerically the infinite integral of equation (E-1) arises in the fact that the integrand is a periodic function of α. The infinite integrals of equation (E-4) can be evaluated in the form of Bessel functions. This is done by writing $\sin n(\rho - \zeta)$ and $\cos n(\rho - \zeta)$ in terms of each angle and noting that the resulting integral is in the form of either a Fourier cosine or sine transform whose value is known.[39] The Bessel function equivalent of the infinite integrals which arise are

$$\int_0^{\infty}\frac{\sin n\rho}{(\rho^2+\sigma^2)^{1/2}}\,d\rho = \frac{1}{2}\pi\left[I_0(n\sigma) - \mathbf{L}_0(n\sigma)\right] \qquad \text{(E-5)}$$

$$\int_0^{\infty} \frac{\cos n\rho}{(\rho^2+\sigma^2)^{1/2}} d\rho = K_o(n\sigma) \tag{E-6}$$

$$\int_0^{\infty} \frac{\sin n\rho}{(\rho^2+\sigma^2)^{3/2}} d\rho = -\frac{\pi}{2}\frac{n}{\sigma}\left[I_1(n\sigma) - \mathbf{L}_{-1}(n\sigma)\right] \tag{E-7}$$

$$\int_0^{\infty} \frac{\cos n\rho}{(\rho^2+\sigma^2)^{3/2}} d\rho = \frac{n}{\sigma} K_1(n\sigma \tag{E-8}$$

The functions $K_o(n\sigma)$ and $K_1(n\sigma)$ are modified Bessel functions[40] of the second kind. $K_o(n\sigma)$ has a logarithmic singularity at $\sigma = 0$ and $K_1(n\sigma)$ has a singularity like $\lim_{\sigma \to 0} \frac{1}{\sigma}$. The functions $I_o(n\sigma)$ and $I_1(n\sigma)$ are modified Bessel functions of the first kind. $I_o(n\sigma)$ has the value one at $\sigma = 0$ and $I_1(n\sigma)$ has the value zero. The functions $\mathbf{L}_o(n\sigma)$ and $\mathbf{L}_1(n\sigma)$ are modified Struve functions; and both are zero at $\sigma = 0$. For these identities to hold, σ must be positive.

After integrating by parts and using the above values for the infinite integrals, the following equation is obtained for the integral I_1.

$$I_1 = I_a + I_b + I_c + I_d \tag{E-9}$$

where

$$I_a = -\frac{4n}{\bar{\chi}^2 \tan^2\beta_o} \int_0^{\pi/2} (\cos 2\phi \cos 2n\phi) \left(\int_0^{\zeta} \frac{\cos n(\rho-\xi)}{(\rho^2+\sigma^2)^{1/2}} d\rho \right) d\phi \tag{E-10}$$

$$I_b = \frac{1}{\bar{x}^2 \tan^2\beta_0} \int_0^{\pi/2} \frac{(\sin 2\phi \sin 2n\phi)}{\sigma^2} \left(\frac{\zeta}{\sqrt{\zeta^2+\sigma^2}} + n\int_0^{\zeta} \frac{\rho \sin n(\rho-\zeta)}{(\rho^2+\sigma^2)^{1/2}}\, d\rho \right) d\varphi \tag{E-11}$$

$$I_c = \frac{-4n}{\bar{x}^2 \tan^2\beta_0} \int_0^{\pi/2} (\cos 2\phi \cos 2n\phi) \left| \cos n\zeta\, K_0(n\sigma) + \frac{\pi}{2} \sin n\zeta \left[I_0(n\sigma) - \mathbf{L}_0(n\sigma) \right] \right| d\varphi \tag{E-12}$$

$$I_b = \frac{-4n}{\bar{x}^2 \tan^2\beta_0} \int_0^{\pi/2} \frac{(\sin 2\phi \sin 2n\phi)}{\sigma} \left| \cos n\zeta\, K_1(n\sigma) - \frac{\pi}{2} \sin n\zeta \left[I_1(n\sigma) - \mathbf{L}_{-1}(n\sigma) \right] \right| d\varphi \tag{E-13}$$

The only difficulty which will arise in evaluating these integrals is when $\sigma = 0$ which can only occur when $\bar{x} = 1$ and $\phi = 0$. For $\bar{x} = 1$, $\sigma = \frac{2}{\tan\beta_0} \sin\phi$. The integrands of integrals for I_a and I_c have logarithmic singularities at $\phi = 0$ when $\bar{x} = 1$ which can easily be removed by a change in variable as discussed in section II.4. The integrands of I_b and I_d are of the indeterminate form 0/0 and can easily be shown to have the following value at $\phi = 0$.

Integrand of I_b at $\phi = 0$

$$\lim_{\phi\to 0} \left(\frac{\cos\phi \sin 2n\phi}{\sin\phi} \left[\frac{\zeta}{\sqrt{\zeta^2+\sigma^2}} + n\int^{\zeta} \frac{\rho \sin n(\rho-\zeta)}{(\rho^2+\sigma^2)^{1/2}}\, d\rho \right.\right.$$

$$= 2n \cos n\zeta \tag{E-14}$$

Integrand of I_d at $\phi = 0$

$$\lim_{\phi\to 0} \left[\cos\phi \sin 2n\phi \cos n\zeta\, K_1(n\sigma) \right] = \tan\beta_0 \cos n\zeta \tag{E-15}$$

Of interest is the value of I_1, equation (E-9), for $\zeta = 0$. For this case the integrals I_a and I_b, equation (E-10) and equation (E-11) respectively, are identically zero and equation I_1 reduces to the following very simplified form.

$$\bar{I}_1 = \frac{4n}{\bar{x}^2 \tan^2\beta_0}\left(\int_0^{\pi/2} \cos 2\phi \cos 2n\phi\, K_0(n\sigma)\, d\phi - \int_0^{\pi/2} \sin 2\phi \sin 2n\phi\, K_1(n\sigma)\, d\phi\right), \quad \zeta = 0 \tag{E-16}$$

The integrand of the first integral has a logarithmic singularity at $\bar{x} = 1$, $\phi = 0$ which can easily be removed. The value of the integrand of the second integral at $\bar{x} = 1$, $\phi = 0$ is obtained from equation (E-15) as $\tan\beta_0$.

The integral I_2, equation (E-2), will now be considered. The notation given by equation (E-3) will be introduced and making the change in variable $\phi = (1/2)(\theta + \alpha)$ equation (E-2) becomes

$$\bar{I}_2 = \frac{2}{\beta^2}\int_0^{\infty}\int_{-\pi/2}^{\pi/2}\left(\frac{(\zeta+\alpha)\cos 2\phi \cos 2n\phi \cos n\alpha - \sin 2\phi \sin 2n\phi \sin n\alpha}{[(\zeta+\alpha)^2 + \sigma^2]^{3/2}}\right) d\phi\, d\alpha \tag{E-17}$$

Interchanging the order of integration and making the change in variable $\rho = \zeta + \alpha$, I_2 becomes

$$\bar{I}_2 = \frac{2}{\beta^2}\int_{-\pi/2}^{\pi/2}\left((\cos 2\phi \cos 2n\phi)\int_{\zeta}^{\infty}\frac{\rho \cos n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\, d\rho - (\sin 2\phi \sin 2n\phi)\int_{\zeta}^{\infty}\frac{\sin n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\, d\rho\right) d\phi$$

$$= \frac{2}{\beta^2}\int_{-\pi/2}^{\pi/2}\left((\cos 2\phi \cos 2n\phi\left[\int_0^{\infty}\frac{\rho \cos n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\, d\rho - \int_0^{\zeta}\frac{\rho \cos n(\rho-\zeta)}{(\rho^2+\sigma^2)^{3/2}}\, d\rho\right]\right) d\phi$$

$$-\frac{2}{\beta^2}\int_{-\pi/2}^{\pi/2} \sin 2\phi \sin 2n\phi \left[\int_0^{\infty} \frac{\sin n(\rho-\varsigma)}{(\rho^2+\sigma^2)^{3/2}}\,d\rho - \int_0^{\varsigma} \frac{\sin n(\rho-\varsigma)}{(\rho^2+\sigma^2)^{3/2}}\,d\rho\right] d\phi$$

Using the Bessel function equivalents to the infinite integrals this equation can be reduced to the following

$$I_2 = I_e + I_f + I_g + I_h \tag{E-18}$$

where

$$I_e = \frac{4}{\bar{x}^2 \tan^2\beta_o}\int_0^{\pi/2} (\cos 2\phi \cos 2n\phi)\left[\frac{1}{\sqrt{\sigma^2+\varsigma^2}} + n\int_0^{\varsigma} \frac{\sin n(\rho-\varsigma)}{(\sigma^2+\rho^2)^{1/2}}\,d\rho\right] d\phi \tag{E-19}$$

$$I_f = \frac{-4n}{\bar{x}^2 \tan^2\beta_o}\int_0^{\pi/2} \frac{(\sin 2\phi \sin 2n\phi)}{\sigma^2}\left[\int_0^{\varsigma} \frac{\rho \cos n(\rho-\varsigma)}{(\rho^2+\sigma^2)^{1/2}}\,d\rho\right] d\phi \tag{E-20}$$

$$I_g = \frac{4n}{\bar{x}^2 \tan^2\beta_o}\int_0^{\pi/2} (\cos 2\phi \cos 2n\phi)\left(\sin n\varsigma\, K_o(n\sigma) - \frac{\pi}{2}\cos n\varsigma\left[I_o(n\sigma) - \mathbf{L}_o(n\sigma)\right]\right) d\phi \tag{E-21}$$

$$I_h = \frac{4n}{\bar{x}^2 \tan^2\beta_o}\int_0^{\pi/2} \frac{(\sin 2\phi \sin 2n\phi)}{\sigma}\left(\frac{\pi}{2}\cos n\varsigma\left[I_1(n\sigma) - \mathbf{L}_1(n\sigma)\right] + \sin\varsigma\, K_1(n\sigma)\right) d\phi \tag{E-22}$$

Difficulty may arise in evaluating these integrals when $\sigma = 0$, which can only occur when $\bar{x} = 1$ and $\phi = 0$ at the same time. As before $\sigma = \frac{2}{\tan\beta_o}\sin\phi$ when $x = 1$. No difficulty is encountered in evaluating the integral for I_e when $\sigma = 0$ except if ς is also zero. This possibility will

be discussed later. The integrand of the integral I_g has a logarithmic singularity at $\phi = 0$ which can easily be removed. Both integrals I_f and I_h have integrands of indeterminate form and can easily be shown to have the following value at $\phi = 0$.

Integrand of I_f at $\phi = 0$

$$\lim_{\phi \to 0} \left(\frac{\cos\phi \sin 2n\phi}{|\sin\phi|} \left[\int_0^{\zeta} \frac{\rho \cos n(\rho - \zeta)}{\sqrt{\rho^2 + \cdots}} \, d\rho \right] \right) = 2 \sin n\zeta \tag{E-23}$$

Integrand of I_h at $\phi = 0$

$$\lim_{\phi \to 0} \left| \sin n\zeta \cos\phi \sin 2n\phi \, K_1(n\sigma) \right| = \tan\beta_0 \sin n\zeta \tag{E-24}$$

The value of I_2 at $\zeta = 0$ is of interest especially because of the form of I_e. As before the integral I_f is identically zero and consequently for $\zeta = 0$, I_2 becomes

$$\bar{I}_2 = \frac{4}{\bar{\chi}^2 \tan^2\beta_0} \int_0^{\pi/2} \frac{(\cos 2\phi \cos 2n\phi)}{\sigma} d\phi - \frac{2\pi n}{\bar{\chi}^2 \tan^2\beta_0} \int_0^{\pi/2} (\cos 2\phi \cos 2n\phi) \left[I_0(n\sigma) - L_0(n\sigma) \right] d\phi$$

$$- \frac{2\pi n}{\bar{\chi}^2 \tan^2\beta_0} \int_0^{\pi/2} \frac{(\sin 2\phi \sin 2n\phi)}{\sigma} \left[I_1(n\sigma) - L_{-1}(n\sigma) \right] d\phi \tag{E-25}$$

If in addition $\bar{x} = 1$, the integrand of the last integral is zero at $\phi = 0$, but the integrand of the first integral has a singularity at $\phi = 0$. In fact it can be shown that the integral itself is infinite as $\ln \infty$. Consider the first integral for $\bar{x} = 1$

$$\int_0^{\pi/2} \frac{\cos 2\phi \cos 2n\phi}{|\sin\phi|}\, d\phi - \int_0^{\pi/2} \frac{\cos 2n\phi}{|\sin\phi|}\, d\phi - 2\int_0^{\pi/2} \sin\phi \cos 2n\phi \, d\phi$$

$$= \int_0^{\pi/2} \frac{1}{|\sin\phi|}\, d\phi - 2\int_0^{\pi/2} \left(\frac{\sin^2 n\phi}{|\sin\phi|} - \sin\phi \cos 2n\phi \right) d\phi$$

but

$$\int_0^{\pi/2} \frac{1}{|\sin\phi|}\, d\phi = \int_0^{1} \frac{dx}{x\sqrt{1-x^2}} = \left[-\ln\left(\frac{1+\sqrt{1-x^2}}{x} \right) \right]_0^1 = \ln\infty$$

END